1 IONS

Introduction

A body is said to vibrate if it has a to-and-fro motion. A pendulum swinging on either side of a mean position does so under the action of gravity. When the pendulum swings through the midposition, its centre of mass is at the lowest point and it possesses only kinetic energy. At each extremity of its swing, it has only potential energy. In the absence of any friction, the motion continues indefinitely. It can be shown that if the swings on either side of the mean position are very small, it approximates to simple harmonic motion.

Usually, vibrations are due to elastic forces. Whenever a body is displaced from its equilibrium position, work is done on the elastic constraints of the forces on the body and is stored as stain energy. Now, if the body is released, the internal forces cause the body to move towards its equilibrium position. If the motion is frictionless, the strain energy stored in the body is converted into kinetic energy during the period the body reaches the equilibrium position at which it has maximum kinetic energy. The body passes through the mean position, the kinetic energy is utilised to overcome the elastic forces and is stored in the form of strain energy, and so on.

18.1 DEFINITIONS

(i) ***Free (Natural) Vibrations*** Elastic vibrations in which there are no friction and external forces after the initial release of the body are known as free or natural vibrations.

(ii) ***Damped Vibrations*** When the energy of a vibrating system is gradually dissipated by friction and other resistances, the vibrations are said to be damped. The vibrations gradually cease and the system rests in its equilibrium position.

(iii) ***Forced Vibrations*** When a repeated force continuously acts on a system, the vibrations are said to be forced. The frequency of the vibrations is that of the applied force and is independent of their own natural frequency of vibrations.

(iv) ***Period*** It is the time taken by a motion to repeat itself, and is measured in seconds.

(v) ***Cycle*** It is the motion completed during one time period.

(vi) ***Frequency*** Frequency is the number of cycles of motion completed in one second. It is expressed in hertz (Hz) and is equal to one cycle per second.

(vii) ***Resonance*** When the frequency of the external force is the same as that of the natural frequency of the system, a state of resonance is said to have been reached. Resonance results in large amplitudes of vibrations and this may be dangerous.

...VIBRATIONS

Consider a vibrating body, e.g., a rod, shaft or spring. Figure 18.1 shows a massless shaft, one end of which is fixed and the other end carrying a heavy disc. The system can execute the following types of vibrations.

(i) ***Longitudinal Vibrations*** If the shaft is elongated and shortened so that the same moves up and down resulting in tensile and compressive stresses in the shaft, the vibrations are said to be longitudinal. The different particles of the body move parallel to the axis of the body [Fig.18.1(a)].

(ii) ***Transverse Vibrations*** When the shaft is bent alternately [Fig.18.1(b)] and tensile and compressive stresses due to bending result, the vibrations are said to be transverse. The particles of the body move approximately perpendicular to its axis.

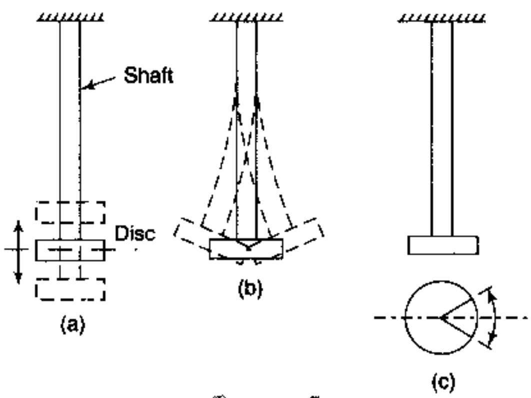


Fig. 18.1

(iii) ***Torsional Vibrations*** When the shaft is twisted and untwisted alternately and torsional shear stresses are induced, the vibrations are known as torsional vibrations. The particles of the body move in a circle about the axis of the shaft [Fig.18.1(c)].

...RES OF VIBRATING SYSTEMS

For mathematical analysis of a vibratory system, it is necessary to have an *idealized model* of the same which appropriately represents the system.

Basic Elements

For a system to vibrate, it must possess inertial and restoring elements whereas it may possess some damping element responsible for dissipating the energy.

Inertial elements These are represented by lumped masses for rectilinear motion and by lumped moment of inertia for angular motion.

Restoring Elements Massless linear or torsional springs represent the restoring elements for rectilinear and torsional motions respectively.

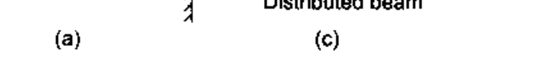


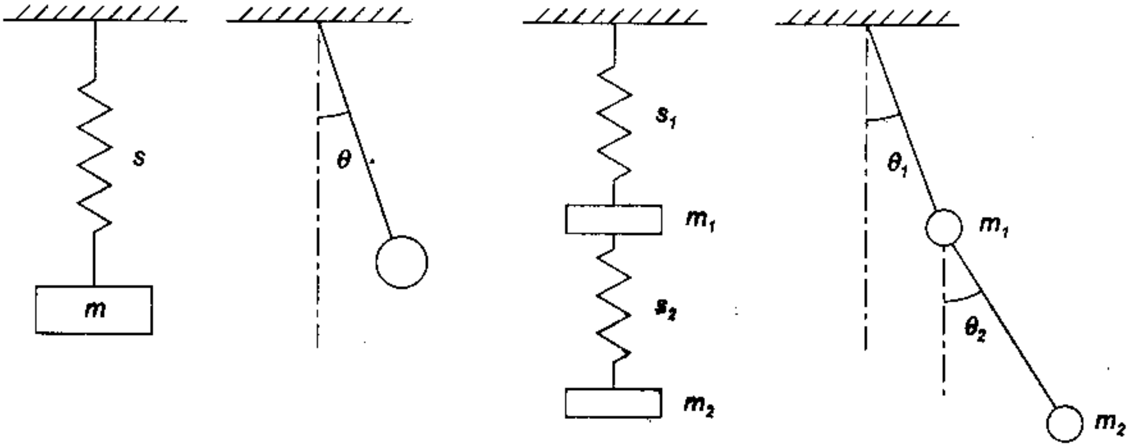
Fig. 18.2

Damping Elements Massless dampers of rigid elements may be considered for energy dissipation in a system.

It is to be noted that lumping of quantities depends upon the distribution of these quantities in the systems. In a spring-mass vibrating system, the spring can be considered massless only if its mass is very less as compared to the suspended mass [Fig.18.2 (a)]. Similarly, if the mass of the beam is negligible as compared to the end mass, lumping is possible [Fig.18.2 (b)], otherwise not [Fig.18.2 (c)].

OF FREEDOM

The number of independent coordinates required to describe a vibratory system is known as its *degree of freedom*. A simple spring-mass system [Fig.18.3 (a)] or a simple pendulum oscillating in one plane [Fig.18.3 (b)] are the examples of single-degree-of-freedom systems. A two-mass, two-spring system constrained to move in one direction [Fig.18.3 (c)], or a double pendulum [Fig.18.3(d)] belong to two-degree-of-freedom systems. A system which has continuously distributed mass such as a string stretched between two supports has infinite degrees of freedom. As such, a system is equivalent to an infinite number of masses concentrated at different points (Fig.18.4).



(a) (b) (c) (d)

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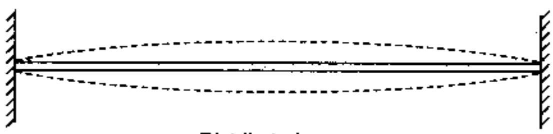

Distributed mass

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In the following sections, different types of vibrations have been discussed separately.

SECTION-1 (LONGITUDINAL VIBRATIONS)

LONGITUDINAL VIBRATIONS

The natural frequency of a vibrating system may be found by any of the following methods.

1. Equilibrium Method

It is based on the principle that whenever a vibratory system is in equilibrium, the algebraic sum of forces and moments acting on it is zero. This is in accordance with D′ Alembert's principle that the sum of the inertia forces and the external forces on a body in equilibrium must be zero.

Figure 18.5(a) shows a helical spring suspended vertically from a rigid support with its free end at *A-A*.

If a mass m is suspended from the free end, the spring is stretched by a distance Δ and *B-B* becomes the equilibrium position [Fig.18.5(b)]. Thus Δ is the static deflection of the spring under the weight of the mass m.

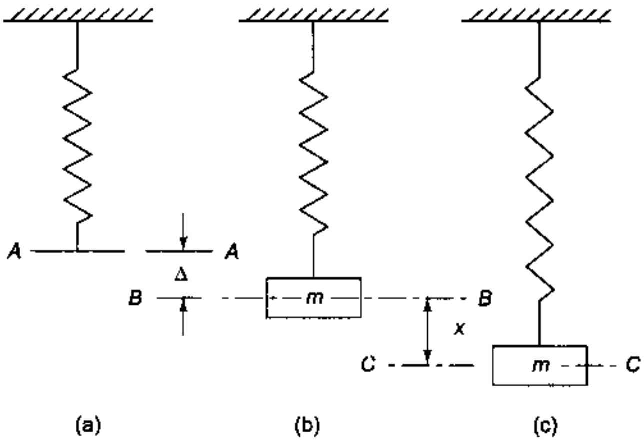


Fig. 18.5

Let s = stiffness of the spring under the weight of the mass m.

In the static equilibrium position,

upward force = downward force

$$s \times \Delta = mg \tag{18.1}$$

Now, if the mass m is pulled farther down through a distance x [Fig.18.5(c)], the forces acting on the mass will be

inertia force $= m\ddot{x}$ (upwards)

spring force (restoring force) $= sx$ (upwards)

(x is downward and thus velocity $\dot{x}$ and acceleration $\ddot{x}$ are also downwards)

As the sum of the inertia force and the external force on the body in any direction is to be zero (D′Alembert's principle),

$$m\ddot{x} + sx = 0 \tag{18.2}$$

If the mass is released, it will start oscillating above and below the equilibrium position. The oscillation will continue for ever if there is no frictional resistance to the motion.

The above equation can be written as

$$\ddot{x} + \left(\frac{s}{m}\right)x = 0 \tag{18.3}$$

This is the equation of a simple harmonic motion and is analogous to

$$\ddot{x} + \omega_n^2 x = 0 \tag{18.4}$$

The solution of which is given by

$$x = A \sin \omega_n t + B \cos \omega_n t \tag{18.5}$$

where A and B are the constants of integration and their values depend upon the manner in which the vibration starts. By making proper substitutions, other forms of the solution can also be obtained as follows:

- By assuming $A = X \cos \varphi$ and $B = X \sin \varphi$,

$$x = X(\sin \omega_n t \cos \varphi + \cos \omega_n t \sin \varphi)$$

$$x = X \sin(\omega_n t + \varphi) \tag{18.6}$$

where X and φ are the constants and have to be found from initial conditions.

- By assuming $A = X \sin \psi$ and $B = X \cos \psi$,

$$x = X(\sin \omega_n t \sin \psi + \cos \omega_n t \cos \psi)$$

$$x = X \cos(\omega_n t - \psi) \tag{18.7}$$

where X and ψ are the constants and have to be found from initial conditions.

The above solutions show that the system vibrates with frequency

$$\omega_n = \sqrt{\frac{s}{m}} \tag{18.8}$$

which is known as the *natural circular frequency* of vibration.

As one cycle of motion is completed in an angle 2π, the period of vibration is

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{s}} \tag{18.9}$$

and *natural linear frequency* of the vibrating system,

$$f_n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} \tag{18.10}$$

In general, the words 'circular' or 'linear' are not used in *natural circular frequency* or in *natural linear frequency*. Both are known as *natural frequencies* of vibration and are distinguished by their units.

Now let us consider different manners of starting the motion.

(i) If the motion is started by displacing the mass through a distance x_o and giving a velocity v_o then for the solution of Eq. 18.5,

$$t = 0, \quad x = x_o \quad \text{and} \quad \dot{x} = v_o$$

and the constants A and B can be found as below:

$$x_o = A(0) + B(1) \quad \text{or} \quad B = x_o$$

Taking the time derivative of Eq. 18.5,

$$\dot{x} = A\,\omega_n \cos\omega_n t - B\,\omega_n \sin \omega_n t \tag{18.11}$$

Thus, $$v_o = A\,\omega_n(1) - B\,\omega_n(0) \quad \text{or} \quad A = \frac{v_o}{\omega_n}$$

Thus Eq. (i) can be written as

$$x = \frac{v_o}{\omega_n} \sin \omega_n t + x_o \cos \omega_n t \tag{18.12}$$

which is the general form of the solution.

The solution is represented graphically in Fig. 18.6.

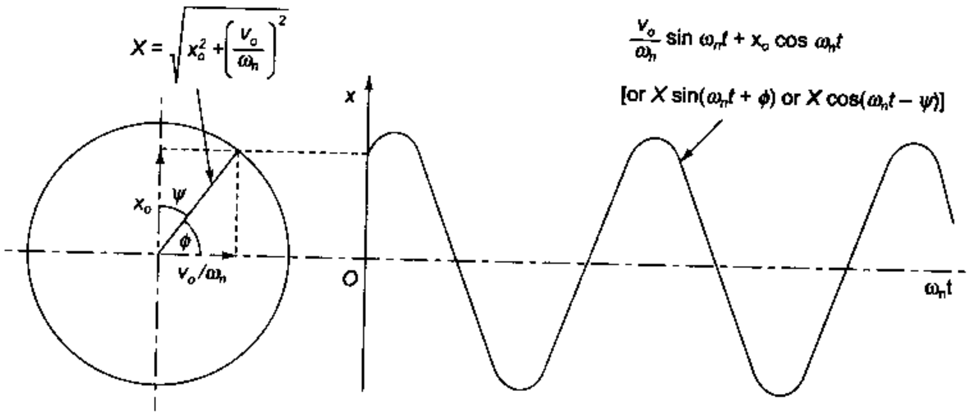


Fig. 18.6

- For the solution of Eq. 18.6 with the same initial conditions, we have

$$x = X \sin (\omega_n t + \varphi) \quad \text{(i)}$$

Taking the time derivative of Eq. 18.6,

$$\dot{x} = X \omega_n \cos(\omega_n t + \varphi) \quad (18.13)$$

$$v_o = X\omega_n \cos \varphi \quad \text{or} \quad \frac{v_o}{\omega_n} = X \cos \varphi \quad \text{(ii)}$$

Squaring and adding (i) and (ii),

$$X = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_n}\right)^2}$$

Dividing (i) by (ii)

$$\tan \phi = \frac{x_o \omega_n}{v_o} \quad \text{or} \quad \phi = \tan^{-1} \frac{x_o \omega_n}{v_o}$$

Thus the equation can be written as

$$x = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_n}\right)^2} \sin (\omega_n t + \varphi) \quad (18.14)$$

The solution is represented graphically in Fig. 18.6.

- In a similar way, the solution of Eq. 18.7 can be written as

$$x = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_n}\right)^2} \cos (\omega_n t - \psi) \quad (18.15)$$

where ψ is given by, $\psi = \tan^{-1} \frac{v_o}{x_o \omega_n}$

The solution is represented graphically in Fig. 18.6.

(ii) If the motion is started by displacing the mass through a distance x_o and then releasing it then at

$t = 0, \quad x = x_o \quad$ and $\quad \dot{x} = 0$

Thus from Eqs 18.5 and 18.11,

$$x_o = A(0) + B(1) \quad \text{or} \quad B = x_o$$

and $\quad 0 = A\,\omega_n(1) - B\,\omega_n(0) \quad$ or $\quad A = 0$

The equation of motion

$$x = x_o \cos \omega_n t \tag{18.16}$$

The solution is represented graphically in Fig. 18.7.

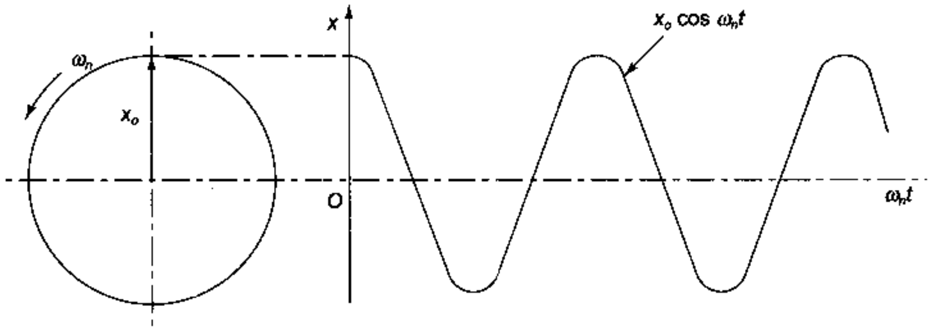


Fig. 18.7

- For the solution of Eq. 18.6, for the same initial conditions,

  $x_o = X \sin \varphi$ (i)

  $0 = X\omega_n \cos \varphi$ (ii) (from Eq. 18.13)

  or $\cos \phi = 0$ (X and ω_n cannot be zero)

  or $\phi = 90°$

  Therefore from (i), $X = x_o$

  and the equation of motion,

  $$x = X \sin(\omega_n t + 90°)$$

  or $$x = x_o \cos \omega_n t$$

  i.e., the same equation as Eq. 18.16.

- For the solution of Eq. 18.7 and for the same initial conditions, the equation of motion can be obtained which will be same as Eq. 18.16.

(iii) If the motion is started by providing a velocity of v_o at the equilibrium position then at

$t = 0, \quad x = 0 \quad$ and $\quad \dot{x} = v_o$

Then constants can be found as before from Eqs 18.5 and 18.11, i.e.,

$0 = A(0) + B(1) \quad$ or $\quad B = 0$

and $\quad v_o = A\,\omega_n(1) - B\,\omega_n(0) \quad$ or $\quad A = \dfrac{v_o}{\omega_n}$

The equation of motion

$$x = \frac{v_o}{\omega_n} \sin \omega_n t \tag{18.17}$$

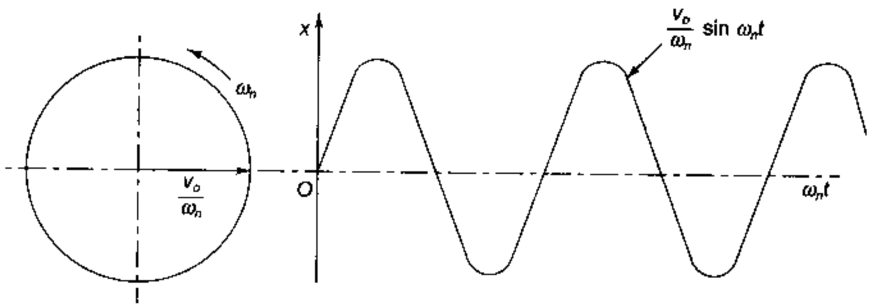


Fig. 18.8

The solution is represented graphically in Fig. 18.8.

- For the sol of Eq. 18.6 and for the same initial conditions

$$0 = X \sin \varphi \quad \text{or} \quad \phi = 0^\circ$$

and $\quad \dot{x} = X\,\omega_n \cos(\omega_n t + \varphi)$

or $\quad v_o = X\,\omega_n \qquad (\phi = 0)$

or $\quad X = \dfrac{v_o}{\omega_n}$

Therefore, equation of motion, $x = \dfrac{v_o}{\omega_n} \sin \omega_n t$

which is the same equation as Eq. 18.17.

- For the solution of Eq. 18.7 and for the same initial conditions, the equation of motion can be obtained which will be same as Eq. 18.17.

Equation 18.6 is considered a more convenient form of the equation. In this equation, the coefficient is the *amplitude* (maximum displacement) of the vibration. φ is called the *phase angle* and is the angular advance of the vector with respect to the sine function.

Equation 18.7 is also a convenient form of the equation.

2. Energy Method

In a conservative system (a system with no damping), the total mechanical energy, i.e., the sum of the kinetic and the potential energies, remains constant and therefore,

$$\frac{d}{dt}(KE + PE) = 0$$

We have

$$KE = \frac{1}{2} m \dot{x}^2$$

and

$$PE = \text{mean force} \times \text{displacement}$$

$$= \frac{0 + sx}{2} \times x$$

$$= \frac{sx^2}{2}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{sx^2}{2}\right) = 0$$

or $$\frac{1}{2}m \times 2\dot{x}\ddot{x} + \frac{1}{2}s \times 2x\dot{x} = 0$$

or $$m\ddot{x} + sx = 0$$

or $$\omega_n = \sqrt{\frac{s}{m}}$$

3. Rayleigh's Method

In this method, the maximum kinetic energy at the mean position (where potential energy is zero) is made equal to the maximum potential (or strain energy) at the extreme position (where the kinetic energy is zero).

Let the motion be simple harmonic.

Therefore, $x = X \sin \omega_n t$

where X = maximum displacement from the mean position to the extreme position.

$\therefore$ $$\dot{x} = \omega_n X \cos \omega_n t, \qquad \dot{x}_{max} = \omega_n t$$

or KE at mean position = PE at extreme position

i.e. $$\frac{1}{2}m(\omega_n X)^2 = \frac{1}{2}sX^2$$

or $$m\omega_n^2 = s \quad \text{or} \quad \omega_n = \sqrt{\frac{s}{m}}$$

In *vertical vibrating systems*, the system vibrates about the static equilibrium position assumed by the mass after its suspension, i.e., about position *B-B* (Fig. 18.5). In case of *horizontal vibrating systems* (Fig. 18.9), however, the gravity has no effect on its motion and thus the system vibrates about the original equilibrium position.

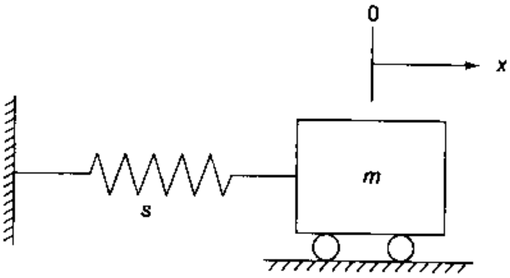


Fig. 18.9

18.6 DISPLACEMENT, VELOCITY AND ACCELERATION

The displacement of the mass m from the mean position at any instant is

$$x = X \sin(\omega_n t + \varphi) \qquad \text{(Eq. 18.6)}$$

Also velocity,

$$v = \dot{x} = X\omega_n \cos(\omega_n t + \varphi)$$
$$= X\omega_n \sin\left[\frac{\pi}{2} + (\omega_n t + \varphi)\right]$$

and acceleration,

$$f = \ddot{x} = -X\omega_n^2 \sin(\omega_n t + \varphi)$$
$$= X\omega_n^2 \sin[\pi + (\omega_n t + \varphi)]$$

These relationships indicate that the velocity vector leads the displacement vector by $\pi/2$ and the acceleration vector leads the displacement vector by π (Fig. 18.10).

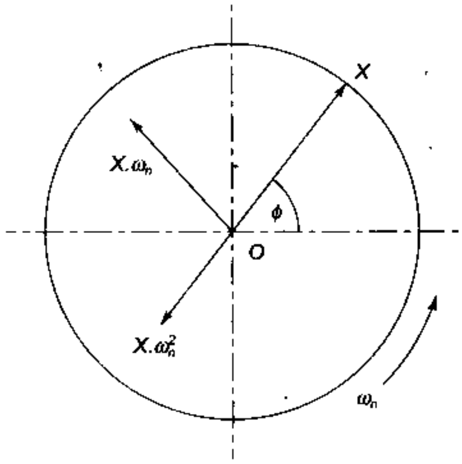


Fig. 18.10

INERTIA EFFECT OF THE MASS OF SPRING

So far, the mass of the spring and thus the effect of inertia have been neglected. The same may be taken into account as follows:

Let m' = mass of the spring wire per unit length

v = velocity of the free end of the spring at the instant under consideration

l = total length of the spring wire

Consider an element of length δy at a length y measured round the coils from the fixed end.

$$KE \text{ of the element} = \frac{1}{2} \times \text{mass of element} \times (\text{velocity of element})^2$$

$$= \frac{1}{2}(m'\delta y) \times \left(\frac{y}{l}v\right)^2$$

$$\text{KE of the spring} = \int_0^l \frac{1}{2} m'v^2 \left(\frac{y}{l}\right)^2 dy$$

$$= \frac{1}{2}\frac{m'v^2}{l^2}\int_0^l y^2 . dy = \frac{1}{2}\frac{m'v^2}{l^2}\frac{l^3}{3}$$

$$= \frac{1}{3}\frac{1}{2}(m'l)v^2$$

$$= \frac{1}{3} \times \left[\frac{1}{2} \times \text{mass of spring} \times (\text{velocity of free end})^2\right]$$

$$= \frac{1}{3} \times KE \text{ of a mass equal to that of the spring moving with the same velocity as the free end}$$

This shows that the inertia effect of the spring is equal to that of a mass one third of the mass of the spring, concentrated at its free end.

Thus

equivalent mass at the free end $= m + \frac{m_1}{3}$

where

m_1 = mass of the spring

$$f_n = \frac{1}{2}\sqrt{\frac{s}{m + \frac{m_1}{3}}} \tag{18.18}$$

It can be noted that the net force on the spring at any instant tending to restore the vibrating mass to the equilibrium position is sx which is proportional to the displacement of the mass. This is true for any vibration due to the elastic forces. Thus in a vibrating system in which the restoring force is proportional to the displacement from the equilibrium position, the frequency of the system will always be given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{mg/\Delta}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} \tag{18.19}$$

where Δ is the static deflection under the suspended mass m.

For example, consider a rod of length l suspended vertically. A mass m is suspended at the free end [Fig. 18.1(a)].

Then

Static deflection, $\Delta = \frac{mgl}{AE}$

where

A = cross-sectional area of the rod

l = length of the rod

E = Young's modulus of the rod material.

Frequency,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{gAE}{mgl}} = \frac{1}{2\pi}\sqrt{\frac{AE}{ml}}$$

However, if the mass of the suspended rod is also to be considered,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{AE}{\left(m + \frac{1}{3}m_1\right)l}}$$

where m_1 = mass of rod

Example 18.1 ***Determine the equivalent spring stiffness and the natural frequency of the following vibrating systems when [refer to Figs 18.11(a) to (e)] the***

(a) mass is suspended to a spring
(b) mass is suspended at the bottom of two springs in series
(c) mass is fixed in between two springs
(d) mass is fixed to the midpoint of a spring
(e) mass is fixed to a point on a bar joining free ends of two springs.

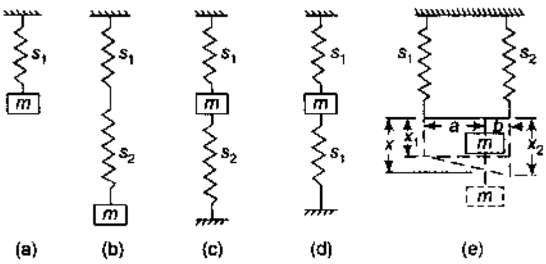


Fig. 18.11

Take

$s_1 = 5$ N/mm, $s_2 = 8$ N/mm
$m = 10$ kg, $a = 20$ mm and $b = 12$ mm

Solution

(a) As there is only one spring, the equivalent spring stiffness is the same, i.e., $s = s_1 = 5$ N/mm

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s_1}{m}} = \frac{1}{2\pi}\sqrt{\frac{5\times 10^3}{10}} = \underline{3.56 \text{ Hz}}$$

(b) Spring force will be the same in the two springs but static deflections will be different.

Let s = equivalent spring stiffness of the two springs.

Deflection of mass m = deflection of Spring 1 + deflection of Spring 2

$$\frac{mg}{s} = \frac{mg}{s_1} + \frac{mg}{s_2}$$

or

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$$

or

$$s = \frac{s_1 s_2}{s_1 + s_2}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{s_1 s_2}{(s_1+s_2)m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{(5\times 10^3)\times(8\times 10^3)}{(5+8)\times 10^3 \times 10}} = \underline{2.79 \text{ Hz}}$$

(c) The spring forces will be different but the deflections will be the same of the two springs and the mass m.

Let Δ = deflection of each spring and of mass m.

Net spring force = spring force in 1 + spring force in 2

$$s\Delta = s_1\Delta + s_2\Delta$$

or $s = s_1 + s_2 = 5 + 8 = 13$ N/mm

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{13\times 10^3}{10}} = \underline{5.74 \text{ Hz}}$$

(d) The spring stiffness of a coiled spring is inversely proportional to the number of coils in the spring. As the mass is fixed at the midpoint, the number of coils becomes half on each side.

Stiffness of spring on each side $= \frac{s_1}{1/2} = 2s_1$

Now the system is similar to case (iii).

Equivalent spring stiffness,

$$s = 2s_1 + 2s_1 = 4s_1$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{4s_1}{m}} = \frac{1}{2\pi}\sqrt{\frac{4\times 5\times 10^3}{10}} = \underline{7.12 \text{ Hz}}$$

(e) Spring forces as well as the static deflections of two springs will be different.

$$\text{Spring force in } 1 = mg\frac{b}{a+b}$$

$$\text{Spring force in } 2 = mg\frac{a}{a+b}$$

$$\text{Deflection of } 1, \Delta_1 = mg\frac{b}{a+b}\frac{1}{s_1}$$

$$\text{Deflection of } 2, \Delta_2 = mg\frac{a}{a+b}\frac{1}{s_2}$$

Assuming that the deflection of 2 is more than that of 1, deflection of mass m,

$$\Delta = \Delta_2 - (\Delta_2 - \Delta_1)\frac{b}{a+b}$$

$$= mg\left[\frac{a}{a+b}\frac{1}{s_2} - \left(\begin{matrix}\frac{a}{a+b}\frac{1}{s_2} \\ -\frac{a}{a+b}\frac{1}{s_1}\end{matrix}\right)\frac{b}{a+b}\right]$$

$$= \frac{mg}{a+b}\left[\frac{a}{s_2} - \frac{ab}{(a+b)s_2} + \frac{b^2}{(a+b)s_1}\right]$$

$$= \frac{mg}{a+b}\left[\frac{a^2 s_1 + abs_1 - abs_1 + b^2 s_2}{(a+b)s_1 s_2}\right]$$

$$= \frac{mg}{(a+b)^2}\left[\frac{a^2}{s_2} + \frac{b^2}{s_1}\right]$$

Total spring force = force in Spring 1 + force in Spring 2

$$s\,\Delta = s_1\,\Delta_1 + s_2\,\Delta_2$$

$$s\frac{mg}{(a+b)^2}\left[\frac{a^2}{s_2} + \frac{b^2}{s_2}\right] = s_1 mg\frac{b}{a+b} \times \frac{1}{s_1} + s_2 mg\frac{a}{a+b} \times \frac{1}{s_2}$$

$$s\frac{1}{a+b}\left(\frac{a^2}{s_2} + \frac{b^2}{s_1}\right) = b + a$$

or

$$s = \frac{(a+b)^2}{\left(\frac{a^2}{s_2} + \frac{b^2}{s_1}\right)}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{(a+b)^2}{\left(\frac{a^2}{s_2} + \frac{b^2}{s_1}\right)m}}$$

$$= \sqrt{\frac{(0.02 + 0.012)^2}{\frac{(0.02)^2}{8\times 10^3} + \frac{(0.012)^2}{5\times 10^3}} \times \frac{1}{10}} = \underline{36\text{ Hz}}$$

Alternatively,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} = \frac{1}{2\pi}\sqrt{\frac{g}{\frac{mg}{(a+b)^2}\left(\frac{a^2}{s_2} + \frac{b^2}{s_1}\right)}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{(a+b)^2}{\left(\frac{a^2}{s_2} + \frac{b^2}{s_1}\right)m}}$$

i.e., the same expression.

Example 18.2 ***Determine the frequency (circular) of vibration of the systems shown in Figs 18.12(a) and (b). Neglect the mass of the pulleys.***

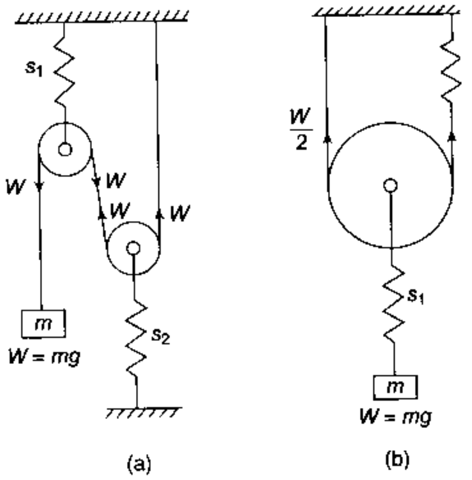


Fig. 18.12

Solution

(a) Force in each spring = $2W$
Deflection on mass m, Δ = 2 (deflection of Spring 1 + deflection of Spring 2)

$$= 2\left(\frac{2W}{s_1} + \frac{2W}{s_2}\right)$$

$$= 4mg\left(\frac{s_1 + s_2}{s_1 s_2}\right)$$

$$\omega_n = \frac{g}{\Delta} = \sqrt{\frac{g(s_1 s_2)}{4mg(s_1 + s_2)}} = \sqrt{\frac{s_1 s_2}{4(s_1 + s_2)m}}$$

(b) Force in Spring 1 = W
Force in Spring 2 = $W/2$
Deflection of mass = deflection of Spring 1 + deflection of Spring 2

$$= \frac{W}{s_1} + \frac{1}{2}\frac{W/2}{s_2}$$

$$= mg\left(\frac{1}{s_1} + \frac{1}{4s_2}\right)$$

$$= mg\left(\frac{4s_2 + s_1}{4s_1 s_2}\right)$$

$$\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{4s_1 s_2}{(s_1 + 4s_2)m}}$$

Example 18.3 *Determine the equation of vibration of the water column in a U-tube shown in Fig. 18.13.*

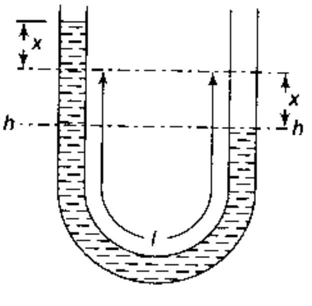


Fig. 18.13

Solution (a) *Newton's Method*

Let a = area of cross section of the tube

ρ = mass density of water

l = total length of water column

Inertia force + External force = 0

Mass × Acceleration + Weight of water column above $h - h = 0$

$$(al\rho) \times \ddot{x} + (a \times 2x)\rho g = 0$$

or

$$\ddot{x} + \frac{2g}{l}x = 0$$

Energy Method At any instant,

$$\frac{d}{dt}(KE + PE) = 0$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(al\rho)\dot{x}^2$$

PE = Work to transfer a water column of length x from the right-hand side to the left-hand side.

$$= mg\,x$$
$$= (a\,x\,\rho)\,g\,x$$
$$= a\,\rho\,g\,x^2$$

$$\frac{d}{dt}\left(\frac{1}{2}al\,\rho\,\dot{x}^2 + a\,\rho\,g\,x^2\right) = 0$$

$$\frac{1}{2}\,a\,l\,\rho \times 2\,\dot{x}\ddot{x} + a\,\rho\,g \times 2\,x\dot{x} = 0$$

$$\ddot{x} + \frac{2g}{l}x = 0 \quad \omega_n = \sqrt{\frac{2g}{l}}$$

Example 18.4 *Determine the natural frequency of a vibrating system shown in Fig. 18.14.*

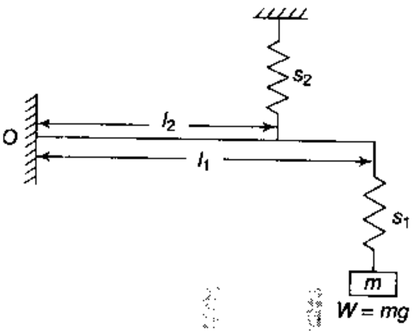


Fig. 18.14

Solution Force in spring 1, $F_1 = W$

Force in spring 2, $F_2 = W\frac{l_1}{l_2}$ $(F_1 \times l_1 = F_2 \times l_2)$

Deflection of mass = deflection of Spring 1 + $\frac{l_1}{l_2}$ (deflection of Spring 2)

$$\Delta = \frac{W}{s_1} + \frac{l_1}{l_2} \times \frac{(Wl_1/l_2)}{s_2} = W\left[\frac{l}{s_1} + \frac{(l_1/l_2)^2}{s_2}\right]$$

$$= mg\left[\frac{s_2 + s_1(l_1/l_2)^2}{s_1 s_2}\right]$$

$$\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{s_1 s_2}{[s_1(l_1/l_2)^2 + s_2]m}}$$

$$= \sqrt{\frac{s_1 s_2 (l_2/l_1)^2}{[s_1 + s_2(l_2/l_1)^2]m}}$$

18.8 DAMPED VIBRATIONS

When an elastic body is set in vibratory motion, the vibrations die out after some time due to the internal molecular friction of the mass of the body and the friction of the medium in which it vibrates.

The diminishing of vibrations with time is called *damping*. External damping can be increased by using dashpots or dampers. A dashpot has a piston which moves in a cylinder filled with some fluid. Shock absorbers, fitted in the suspension system of a motor vehicle, reduce the movement of the springs when there are sudden shocks, thus damping out the bouncing which could have occurred otherwise.

As before, consider a helical spring suspended from a fixed support (Fig. 18.15). *A-A* is the level of the free end before the mass *m* is suspended. *B-B* is the level of static equilibrium under the weight of the mass. The mass is attached to a dashpot to retard its movement.

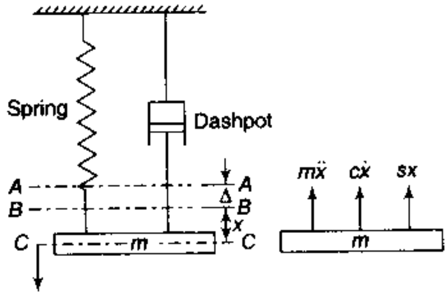


Fig. 18.15

It is usual to assume that the damping force is proportional to the velocity of vibration at lower values of speed and proportional to the square of the velocity at higher speeds. Only the former case will be considered in this chapter.

Consider the forces on the mass *m* when it is displaced through a distance below the equilibrium position during vibratory motion.

Let s = stiffness of the spring

c = damping coefficient (damping force per unit velocity)

ω_n = frequency of natural undamped vibrations

x = displacement of mass from mean position at time t

$v = \dot{x}$ = velocity of the mass at time t

$f = \ddot{x}$ = acceleration of the mass at time t

When the mass moves downwards, the friction force of the dashpot acts in the upward direction.

Now, the forces acting on the mass are

- Inertia $= m\ddot{x}$ (upwards)
- Damping force $= c\dot{x}$ (upwards)
- Spring force (restoring force) $= sx$ (upwards)

As the sum of the inertia force and the external forces on a body in any direction is to be zero,

$$m\ddot{x} + c\dot{x} + sx = 0$$

or

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{s}{m}x = 0 \tag{18.20}$$

It is a differential equation of the second order. Its solution will be of the form

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t} \tag{18.21}$$

where A and B are some constants, α_1 and α_2 are the roots of the auxiliary equation

$$\alpha^2 + \frac{c}{m}\alpha + \frac{s}{m} = 0 \tag{18.22}$$

i.e.,

$$\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)} \tag{18.23}$$

The ratio of $\left(\frac{c}{2m}\right)^2$ to $\left(\frac{s}{m}\right)$ represents the degree of dampness provided in the system and its square root is known as *damping factor* or *damping ratio* ζ, i.e.

$$\zeta = \sqrt{\frac{(c/2m)^2}{s/m}} = \frac{c}{2\sqrt{sm}}$$

or

damping coefficient,

$$c = 2\zeta\sqrt{sm} = 2\zeta m\omega_n = 2\zeta\frac{s}{\omega_n} \tag{18.24}$$

When $\zeta = 1$, the damping is known as critical. The corresponding value of damping coefficient c is denoted by c_c.

Thus under critical damping conditions,

$$c = 2\sqrt{sm} = 2m\omega_n = 2s/\omega_n \tag{18.25}$$

and

$$\zeta = \frac{c}{c_c} = \frac{\text{Actual damping coefficient}}{\text{Critical damping coefficient}} \tag{18.26}$$

Thus when

$\zeta = 1$, the damping is critical

$\zeta > 1$, the system is over-damped

$\zeta < 1$, the system is under-damped

Equation (18.20) can also be written as

$$\ddot{x} + 2\zeta\,\omega_n\dot{x} + \omega_n^2 x = 0 \tag{18.27}$$

and

$$\alpha_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

The exact solution of Eq. (18.27) will depend upon whether the roots $\alpha_{1,2}$ are real or imaginary.

(i) $\zeta > 1$, i.e., the system is over-damped.

The roots of the auxiliary equation are real.

$$\alpha_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

Therefore, the solution is

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

$$= Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \tag{18.28}$$

Constants A and B can be determined from the initial conditions. This is the equation of an aperiodic motion, i.e., the system cannot vibrate due to over-damping. The magnitude of the resultant displacement approaches zero with time.

(ii) $\zeta < 1$, i.e., the system is underdamped.

The roots of the auxiliary equation are imaginary.

$$\alpha_{1,2} = (-\zeta \pm i\sqrt{1-\zeta^2})\omega_n$$

$$\therefore \quad x = Ae^{(-\zeta+i\sqrt{1-\zeta^2})\omega_n t} + Be^{(-\zeta-i\sqrt{1-\zeta^2})\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left[Ae^{(i\sqrt{1-\zeta^2})\omega_n t} + Be^{(-i\sqrt{1-\zeta^2})\omega_n t}\right]$$

Put

$$\sqrt{1-\zeta^2}\,\omega_n = \omega_d$$

Then

$$x = e^{-\zeta\omega_n t}[Ae^{i\omega_d t} + Be^{-i\omega_d t}]$$

$$= e^{-\zeta\omega_n t}[A(\cos\omega_d t + i\sin\omega_d t) + B(\cos\omega_d t - i\sin\omega_d t)]$$

$$= e^{-\zeta\omega_n t}[(A+B)\cos\omega_d t + i(A-B)\sin\omega_d t]$$

$$= e^{-\zeta\omega_n t}[C\cos\omega_d t + D\sin\omega_d t] \tag{18.29}$$

where

$$C = A + B \quad \text{and} \quad D = i\,(A - B)$$

Constants C and D can be found from initial conditions. Alternatively, put

$$A + B = X\sin\varphi \quad \text{and} \quad i\,(A - B) = X\cos\varphi$$

Thus

$$x = e^{-\zeta\omega_n t}(X\sin\varphi\cos\omega_d t + X\cos\varphi\sin\omega_d t)$$

$$= Xe^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) \tag{18.30}$$

Constants X and φ are to be determined from initial conditions. This equation indicates that the system oscillates with frequency $\omega_d (= \sqrt{1-\zeta^2}\,\omega_n)$. As ζ is less than 1, ω_d is always less than ω_n.

The solution consists of three terms:

- X, which is constant
- $e^{-\zeta\omega_n t}$, which decreases with time and finally $e^{-\infty} = 0$
- $\sin(\omega_d t + \varphi)$ which represents a repetition of motion

Thus, the resultant motion is oscillatory with decreasing amplitudes having a frequency of ω_d. Ultimately, the motion dies down with time.

Also,

$$\text{linear frequency, } f_d = \frac{\omega_d}{2\pi}$$

and

$$\text{time period, } T_d = \frac{\omega_d}{2\pi}$$

let $\quad X_0$ = displacement at the start of motion when $t = 0$

X_1 = displacement at the end of first oscillation when $t = T_d$

$$= Xe^{-\zeta\omega_n T_d} \sin(\omega_d T_d + \varphi)$$

$$= Xe^{-\zeta\omega_n T_d} \sin\left(\omega_d \frac{2\pi}{\omega_d} + \varphi\right)$$

$$= Xe^{-\zeta\omega_n T_d} \sin\varphi$$

X_2 = displacement at the end of second oscillation

$$= Xe^{-\zeta\omega_n \times 2T_d} \sin\varphi$$

Similarly,

$$X_3 = Xe^{-\zeta\omega_n \times 3T_d} \sin\varphi$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_n = Xe^{-\zeta\omega_n \times nT_d} \sin\varphi$$

$$X_{n+1} = Xe^{-\zeta\omega_n \times (n+1)T_d} \sin\varphi$$

Then

$$\frac{X_n}{X_{n+1}} = e^{\zeta\omega_n T_d} = \frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = \ldots \tag{18.31}$$

which shows that the ratio of amplitudes of two successive oscillations is constant (Fig. 18.16).

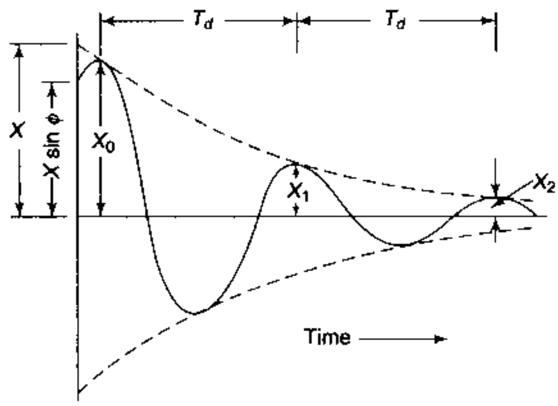


Fig. 18.16

(iii) $\zeta = 1$, i.e., the damping is critical.

The roots of the auxiliary equation are equal, each being equal to $-\omega_n$ and the solution is

$$x = (A + Bt)\, e^{-\omega_n t} \tag{18.32}$$

Since $e^{-\omega_n t}$ approaches zero as $t \to \infty$, the motion is aperiodic. The displacement will be approaching to zero with time.

Figure 18.17 shows the characteristics of motion for the three different cases discussed. The diagram shows that in a critically damped system, the displaced mass returns to the position of rest in the shortest possible time without oscillation. Due to this reason, large guns are critically damped so that they return to their original position (after recoiling because of firing) in the minimum possible time. If the gun barrels are over-damped, they will take more time to return to their original positions.

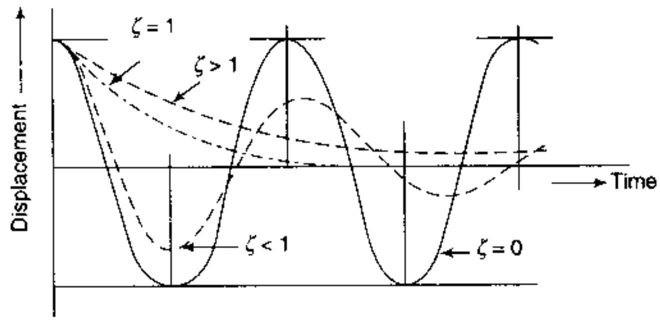


Fig. 18.17

The following points can be noted:

(i) An undamped system ($\zeta = 0$) vibrates at its frequency which depends upon the static deflection under the weight of its mass ($\omega_n = \sqrt{g/\Delta}$).

(ii) When the system is underdamped ($\zeta < 1$), the frequency of the system decreases to $\omega_d (= \sqrt{1-\zeta^2}\,\omega_n)$ and the time period increases to $T_d = 2\pi/\omega_d$. The amplitudes of the vibrations decrease with time, the ratio of successive amplitudes being constant. The vibrations die down with time.

(iii) At critical damping, $\zeta = 1$, $\omega_d = 0$ and $T_d = \infty$. The system does not vibrate and the mass m moves back slowly to the equilibrium position.

(iv) For an overdamped system, $\zeta > 1$, the system behaves in the same manner as for critical damping.

(v) ζ is the ratio of the existing damping in a system to that required for critical damping, i.e., $\zeta = c/c_c$.

18.9 LOGARITHMIC DECREMENT

In Section 18.7, it was observed that the ratio of two successive oscillations is constant in an underdamped system. Natural logarithm of this ratio is called logarithmic decrement and is denoted by

$$\delta = \ln\left(\frac{X_n}{X_{n+1}}\right) = \ln e^{(\zeta\omega_n T_d)} = \zeta\omega_n T_d$$

or

$$\delta = \zeta\omega_n \frac{2\pi}{\omega_d} = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{18.33}$$

Example 18.5 *A vibrating system consists of a mass of 50 kg, a spring with a stiffness of 30 kN/m and a damper. The damping provided is only 20% of the critical value. Determine the*

(i) damping factor
(ii) critical damping coefficient
(iii) natural frequency of damped vibrations
(iv) logarithmic decrement
(v) ratio of two consecutive amplitudes

Solution

$m = 50$ kg $\quad s = 30\ 000$ N/m $\quad c = 0.2\ c_c$

(i) $\zeta = \dfrac{c}{c_c} = \underline{0.2}$

(ii) $c_c = 2\sqrt{sm} = 2\sqrt{30\ 000 \times 50} = 2450$ N/m/s $= \underline{2.45 \text{ N/mm/s}}$

(iii) $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

where

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{30\ 000}{50}} = 24.5 \text{ rad/s}$$

$$\omega_d = \sqrt{1-(0.2)^2} \times 24.5 = \underline{24 \text{ rad/s}}$$

(iv) $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \dfrac{2\pi \times 0.2}{\sqrt{1-(0.2)^2}} = \underline{1.28}$

(v) $\dfrac{X_n}{X_{n+1}} = e^{\delta} = e^{1.28} = \underline{3.6}$

Example 18.6 *Determine the time in which the mass in a damped vibrating system would settle down to 1/50 th of its initial deflection for the following data:*

m = 200 kg $\quad \zeta = 0.22$ s = 40 N/mm

Also, find the number of oscillations completed to reach this value of deflection.

Solution We know

$$\frac{X_0}{X_N} = e^{\zeta\omega_n NT_d}$$

where

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{40 \times 10^3}{200}} = 14.14 \text{ rad/s}$$

$\therefore \quad 50 = e^{0.22 \times 14.14 NT_d}$

or

Total time $NT_d = \underline{1.26 \text{ s}}$

$$T_d = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$= \frac{2\pi}{(\sqrt{1-(0.22)^2}) \times 14.14} = 0.455 \text{ s}$$

Number of oscillations completed $= \dfrac{1.26}{0.455} = \underline{2.76}$

Example 18.7 *In a single-degree damped vibrating system, a suspended mass of 8 kg makes 30 oscillations in 18 seconds. The amplitude decreases to 0.25 of the initial value after 5 oscillations. Determine the*

(i) stiffness of the spring
(ii) logarithmic decrement
(iii) damping factor, and
(iv) damping coefficient

Solution

$m = 8$ kg, $N = 30$, $t = 18$s

$f_n = \dfrac{30}{18} = 1.67$ Hz

$\omega_n = 2\pi f_n = 2\pi \times 1.67 = 10.47$ rad/s

(i) $\omega_n = \sqrt{\dfrac{s}{m}}$

$$10.47 = \sqrt{\frac{s}{8}}$$

$\therefore \quad s = 877$ N/m $\quad$ or $\quad \underline{0.877 \text{ N/mm}}$

(ii) $\dfrac{X_0}{X_5} = \dfrac{X_0}{X_1} \times \dfrac{X_1}{X_2} \times \dfrac{X_2}{X_3} \times \dfrac{X_3}{X_4} \times \dfrac{X_4}{X_5}$

$$= \left(\frac{X_0}{X_1}\right)^5 \cdots \left(\frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = \frac{X_3}{X_4} = \frac{X_4}{X_5}\right)$$

$$\therefore \quad \left(\frac{X_0}{X_1}\right) = \left(\frac{X_0}{X_5}\right)^{1/5} = \left(\frac{1}{0.25}\right)^{1/5} = 1.32$$

$$\delta = \ln\left(\frac{X_0}{X_5}\right) = \ln 1.32 = 0.278$$

(iii) $\dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 0.278$

or

$$\sqrt{1-\zeta^2} = 22.6\zeta$$
$$1-\zeta^2 = 510.82\zeta^2$$
$$\zeta^2 = 0.00195$$
$$\zeta = \underline{0.0442}$$

(iv) $c = 2\,m\,\omega_n\,\zeta$

$$= 2 \times 8 \times 10.47 \times 0.0442$$
$$= \underline{7.4 \text{ N/m/s}}$$

Example 18.8 *A machine mounted on springs and fitted with a dashpot has a mass of 60 kg. There are three springs, each of stiffness 12 N/mm. The amplitude of vibrations reduces from 45 to 8 mm in two complete oscillations. Assuming that the damping force varies as the velocity, determine the*

(i) damping coefficient

(ii) ratio of frequencies of damped and undamped vibrations

(iii) periodic time of damped vibrations

Solution $m = 60$ kg

Stiffness of each spring = 12 N/mm

Combined stiffness, $s = 12 \times 3 = 36$ N/mm

$= 36 \times 10^3$ N/m

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{36\times10^3}{60}} = 24.49 \text{ rad/s}$$

(i) $\dfrac{X_0}{X_2} = \dfrac{X_0}{X_1} \times \dfrac{X_1}{X_2}$

$$= \left(\frac{X_0}{X_1}\right)^2 \qquad \left(\frac{X_1}{X_2} = \frac{X_0}{X_1}\right)$$

or

$$\left(\frac{X_0}{X_1}\right) = \left(\frac{X_0}{X_2}\right)^{1/2} = \left(\frac{45}{8}\right)^{1/2} = 2.37$$

$$\frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\ 2.37 = 0.864$$

$$1-\zeta^2 = 52.88\,\zeta^2$$
$$\zeta^2 = 0.0185$$
$$\zeta = 0.136$$
$$c = 2m\,\omega_n\,\zeta = 2 \times 60 \times 24.49 \times 0.136$$
$$= 400 \text{ N/m/s}$$
$$= \underline{0.4 \text{ N/mm/s}}$$

(ii) $\dfrac{\text{Damped frequency}}{\text{Undamped frequency}} = \dfrac{\omega_d}{\omega_n}$

$$= \frac{\sqrt{1-\zeta^2}\,\omega_n}{\omega_n} = \sqrt{1-\zeta^2}$$

$$= \sqrt{1-(0.136)^2} = \underline{0.99}$$

(iii) $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$

$$= \frac{2\pi}{(\sqrt{1-(0.136)^2}) \times 24.49} = \underline{0.259 \text{ s}}$$

Example 18.9 *A machine weighs 18 kg and is supported on springs and dashpots. The total stiffness of the springs is 12 N/mm and the damping is 0.2 N/mm/s. The system is initially at rest and a velocity of 120 mm/s is imparted to the mass. Determine the*

(i) displacement and velocity of mass as a function of time

(ii) displacement and velocity after 0.4s

Solution

$m = 18$ kg $\quad v = 0.12$ m/s

$s = 12$ N/mm $= 12\ 000$ N/m

$c = 0.2$ N/mm/s $= 200$ N/m/s

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{12\ 000}{18}} = 25.82 \text{ rad/s}$$

$$c = 2\,m\,\omega_n\,\zeta$$
$$200 = 2 \times 18 \times 25.82 \times \zeta$$
$$\zeta = 0.215$$
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$
$$= \sqrt{1-(0.215)^2} \times 25.82 = 25.2 \text{ rad/s}$$

(i) $x = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)$ [Eq. (18.30)]

$x = 0$ at $t = 0$

$\therefore \; X \sin \varphi = 0$

or

$\sin \varphi = 0$ (X cannot be zero)

or

$\varphi = 0$

$\therefore \; x = Xe^{-\zeta\omega_n t} \sin(\omega_d t)$

$\dot{x} = Xe^{-\zeta\omega_n t} \omega_d \cos(\omega_d t)$

$+ X \sin \omega_d t(-\zeta\omega_n)e^{-\zeta\omega_n t}$

$\dot{x} = 0.12$ at $t = 0$

$\therefore \; 0.12 = X\omega_d = 25.2\, X$

or $X = 0.004\ 76$ m $= 4.76$ mm

Displacement, $x = 4.76\, e^{\;0.215 \times 25.82\, t} \sin(25.2t)$

or $x = 4.76\, e^{-5.55t} \sin 25.2\, t$

Velocity,

$\dot{x} = X\, e^{\;\zeta\omega_n t}\, [\omega_d \cos \omega_d t - \zeta\omega_n \sin \omega_d t]$

$= 4.76\, e^{\;5.55\, t}[25.2 \cos 25.2\, t - 5.55 \sin 25.2\, t]$

$= e^{-5.55\, t}[\,120 \cos 25.2\, t - 26.4 \sin 25.2\, t]$

(ii) $x = 4.76\, e^{-5.55 \times 0.4} \sin(25.2 \times 0.4)$

$= 4.76\, e^{-2.22} \sin(10.08 \text{ rad})$

$= 4.76 \times 0.1086 \times (-0.6093)$

$= -0.315$ mm

$\dot{x} = e^{\;5.55 \times 0.4}\, [120 \cos(25.2 \times 0.4) - 26.4 \sin(25.2 \times 0.4)]$

$= 0.1086 \times [120 \cos(10.08 \text{ rad}) - 26.4 \sin(10.08 \text{ rad})]$

$= 0.1086\, [-95.15 - (-16.086)]$

$= -8.587$ mm/s

Example 18.10 ***A gun is so designed that, on firing, the barrel recoils against a spring. A dashpot, at the end of the recoil, allows the barrel to come back to its initial position within the minimum time without any oscillation. The gun barrel has a mass of 500 kg and a recoil spring of 300 N/mm. The barrel recoils 1 m on firing. Determine the***

(i) initial recoil velocity of the gun barrel, and

(ii) critical damping coefficient of the dashpot engaged at the end of the recoil stroke.

Solution

$m = 500$ kg $s = 300$ N/mm $x = 1$ m

(i) The dashpot does not operate during the recoil.

KE of the barrel = Work done on the spring

$$\frac{1}{2} mv^2 = \frac{1}{2} sx^2$$

$$\frac{1}{2} \times 500 \times v^2 = \frac{1}{2} \times (300 \times 10^3) \times (1)^2$$

$v = \underline{24.5 \text{ m/s}}$

(ii) $c_c = 2m\,\omega_n$

But $\omega_n = \sqrt{\dfrac{s}{m}} = \sqrt{\dfrac{300 \times 10^3}{500}} = 24.5$ m/s

$\therefore \; c_c = 2 \times 500 \times 24.5 = 24\ 500$ N/m/s

or $\underline{24.5 \text{ N/mm/s}}$

18.10 FORCED VIBRATIONS

The forcing may be step-input, harmonic or periodic as discussed below:

Step-Input Forcing

Application of a constant force to the mass of a vibrating system is known as step-input forcing. The equation of motion will be

$$m\ddot{x} - sx = F$$

The effect of the constant force F on the system will be similar to the applied weight force due to the mass of the vibrating system (Sec. 18.6, Fig. 18.5) in which the mass vibrates about B-B, i.e., the equilibrium position assumed after the applied weight (force), the displacement being mg/s from the position A-A. In a similar way, on application of the force F, the system will vibrate about the new equilibrium position, the displacement of which will be F/s.

Harmonic Forcing

Consider a mass attached to a helical spring and suspended from a fixed support (no damping). Before the mass is set in motion, let *B-B* be the static equilibrium position under the weight of the mass (Fig. 18.18). Assume now that the mass is subjected to an oscillating force $F = F_o \sin \omega t$, the forces acting on the mass at any instant will be

- Impressed oscillating force $F = F_o \sin \omega t$ (downwards)
- Inertia forces $= m\ddot{x}$ (upwards)
- Spring force (restoring force) $= sx$ (upwards)

Thus the equation of motion will be

$$m\ddot{x} + sx = F_o \sin \omega t \qquad (18.34)$$

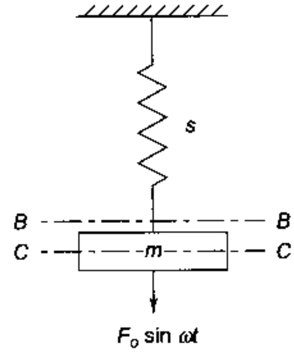


Fig. 18.18

The solution of this equation will consist of the complementary function (*CF*) and the particular integral (*PI*). *CF* is the solution of the equation $m\ddot{x} + sx = 0$ and is

$$CF = X \sin(\omega_n t + \phi)$$

PI can be obtained by using the *D* operator,

$$(D^2 + s/m)\,x = (F_o/m) \cos \omega t$$

$$PI = \frac{(F_o/m)\sin \omega t}{D^2 + (s/m)} = \frac{(F_o/m)\sin \omega t}{-\omega^2 + (s/m)} = \frac{(F_o/m)}{(s/m) - \omega^2} \sin \omega t$$

Multiplying the numerator and denominator by m/s

$$\text{Particular integral} = \frac{F_0/s}{1 - (\omega/\omega_n)^2} \sin \omega t$$

Therefore, the complete solution is

$$x = X \sin(\omega_n t + \phi) + \frac{F_0/s}{1 - (\omega/\omega_n)^2} \sin \omega t \qquad (18.35)$$

Thus the resultant motion is the sum of two harmonics. The constants X and ϕ of the first harmonic are obtained from the initial conditions.

Figure 18.19 shows the motion formed by two phasors of different lengths and rotational velocities.

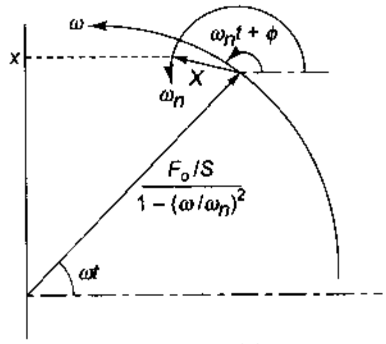


Fig. 18.19

Periodic Forcing

A periodic force is one in which the motion repeats itself in all details after a certain interval of time. It can be shown mathematically that any periodic curve of frequency ω can be represented by a series of harmonic functions, the frequency of each harmonic being an integral multiple of frequency ω, i.e.,

$$f(t) = a_o + a_1 \sin \omega t + a_2 \sin 2\omega t + a_3 \sin 3\omega t + \ldots\; a_n \sin n\omega t + \ldots$$

$$+\, a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \ldots a_n \cos n\omega t + \ldots$$

The series given by this equation is known as *Fourier series*. The various amplitudes $a_1, a_2 \ldots b_1, b_2 \ldots$, etc., of sine and cos waves can be found analytically when $f(t)$ is known. The harmonic of frequency ω is known as the fundamental or the first harmonic of $f(t)$ and the harmonic of frequency $n\omega$, the nth harmonic.

Thus, a periodic force is represented by

$$F(t) = F_o + F_1 \sin \omega t + F_2 \sin 2\omega t + F_3 \sin 3\omega t + \ldots F_n \sin n\omega t + \ldots$$
$$+ F_1 \cos \omega t + F_2 \cos 2\omega t + F_3 \cos 3\omega t + \ldots F_n \cos n\omega t + \ldots$$

and the differential equation of the system becomes

$$m\ddot{x} + sx = F_o + F_1 \sin \omega t + F_2 \sin 2\omega t + F_3 \sin 3\omega t + \ldots F_n \sin n\omega t + \ldots$$
$$+ F_1 \cos \omega t + F_2 \cos 2\omega t + F_3 \cos 3\omega t + \ldots F_n \cos n\omega t + \ldots$$

The response of the complete periodic forcing is the vector sum of the responses to the complimentary functions and particular solutions of the individual forcing functions as on the right-hand side of the equation.

18.11 FORCED-DAMPED VIBRATIONS

A mass m is attached to a helical spring and is suspended from a fixed support as before. Damping is also provided in the system with a dashpot (Fig. 18.20).

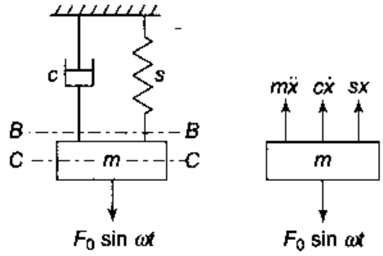


Fig. 18.20

Before the mass is set in motion, let B-B be the static equilibrium position under the weight of the mass. Now, if the mass is subjected to an oscillating force $F = F_0 \sin \omega t$, the forces acting on the mass at any instant will be

- Impressed oscillating force (downwards) $F = F_0 \sin \omega t$
- Inertia force (upwards) $= m\ddot{x}$
- Damping force (upwards) $= c\dot{x}$
- Spring force (restoring force) $= sx$ (upwards)

Thus the equation of motion will be

$$m\ddot{x} + c\dot{x} + sx - F_0 \sin\omega t = 0$$

or
$$m\ddot{x} + c\dot{x} + sx = F_0 \sin\omega t \qquad (18.36)$$

Complete solution of this equation consists of two parts, the complementary function (*CF*) and the particular integral (*PI*).

$$CF = Xe^{-\zeta\omega_n t} \sin(\omega_d t + \varphi_1) \qquad \text{[refer to Eq. (18.30)]}$$

To obtain the *PI*, let

$$\frac{c}{m} = a, \frac{s}{m} = b, \text{ and } \frac{F_0}{m} = d$$

Then, using the operator D, the equation becomes

$$(D^2 + aD + b)\,x = d \sin \omega t$$

$$PI = \frac{d \sin \omega t}{D^2 + aD + b}$$

$$= \frac{d \sin \omega t}{-\omega^2 + aD + b}$$

$$= \frac{1}{(b-\omega^2)+aD} \times \frac{(b-\omega^2)-aD}{(b-\omega^2)-aD} d \sin \omega t$$

$$= d\left[\frac{\sin \omega t(b-\omega^2) - aD \sin \omega t}{(b-\omega^2)^2 - a^2D^2}\right]$$

$$= d\left[\frac{\sin \omega t(b-\omega^2) - a\omega \cos \omega t}{(b-\omega^2)^2 + (a\omega)^2}\right]$$

Take $(b - \omega^2) = R \cos \varphi$ and $a\,\omega = R \sin \varphi$

Constants R and φ are given by

$$R = \sqrt{(b-\omega^2)^2 + (a\omega)^2} \quad \text{and} \quad \varphi = \tan^{-1} \frac{a}{b-\omega^2}$$

$$PI = \frac{dR(\sin \omega t \cos \varphi - \cos \omega t \sin \varphi)}{(b-\omega^2)^2 + (a\omega)^2}$$

$$= \frac{d\sqrt{(b-\omega^2)^2 + (a\omega)^2}}{(b-\omega^2) + (a\omega)^2} \sin(\omega t - \varphi)$$

$$= \frac{d}{\sqrt{(b-\omega^2)^2 + (a\omega)^2}} \sin(\omega t - \varphi)$$

$$= \frac{F_0 / m}{\sqrt{\left(\frac{s}{m} - \omega^2\right)^2 + \left(\frac{c}{m}\omega\right)^2}} \sin(\omega t - \varphi)$$

$$= \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \varphi)$$

$$x = CF - PI$$

$$Xe^{-\zeta\omega_n t} \sin(\omega_d t - \varphi_1) + \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t \qquad (18.37)$$

The damped-free vibrations represented by the first part (CF) becomes negligible with time as $e^{-\infty} = 0$. The steady-state response of the system is then given by the second part PI.

The amplitude of the steady-state response is given by

$$A = \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}} \qquad (18.38)$$

$$= \frac{F_0 / s}{\sqrt{\left(1 - \frac{m\omega^2}{s}\right)^2 + \left(\frac{c}{s}\omega\right)^2}}$$

$$= \frac{F_0 / s}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \qquad (18.39)$$

The equation is in the dimensionless form and is more convenient for analysis. It may be noted that the numerator F_o/s is the static deflection of the spring of stiffness s under a force F_o. The frequency of the steady-state forced vibration is the same as that of the impressed vibrations. φ is the phase lag for the displacement relative to the velocity vector.

$$\tan\varphi = \frac{a\omega}{b-\omega^2} = \frac{\frac{c}{m}\omega}{\frac{s}{m}-\omega^2} = \frac{c\omega}{s-m\omega^2} = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad (18.40)$$

The particular solution of the equation of motion can also be obtained graphically as follows:

Assume that the displacement of the vibrating mass under the action of the applied simple harmonic force $F_o \sin \omega t$ is also simple harmonic and lags by an amount φ. Then

$$x = A\sin(\omega t - \varphi)$$

and

$$\dot{x} = \omega A\cos(\omega t - \varphi) = \omega A \sin\left[\frac{\pi}{2} + (\omega t - \varphi)\right]$$

$$\ddot{x} = -\omega^2 A\sin(\omega t - \varphi)$$

where A is the amplitude of vibrations.

Substituting these values in the equation

$$m\ddot{x} + c\dot{x} + sx = F_o \sin\omega t$$

$$-m\omega^2 A\sin(\omega t-\varphi) + c\omega A\sin\left[\frac{\pi}{2}+(\omega t-\varphi)\right] + sA\sin(\omega t-\varphi) - F_0\sin\omega t = 0$$

$$F_0\sin\omega t + m\omega^2 A\sin(\omega t-\varphi) - c\omega A\sin\left[\frac{\pi}{2}+(\omega t-\varphi)\right] - sA\sin(\omega t-\varphi) = 0$$

The forces and the vector sum of the same have been shown in Fig. 18.21. In triangle abc,

$$\sqrt{(sA - m\omega^2 A)^2 + (c\omega A)^2} = F_0$$

or

$$A\sqrt{(s-m\omega^2) + (c\omega)^2} = F_0$$

or

$$A = \frac{F_0}{\sqrt{(s-m\omega^2)^2 + (c\omega)^2}}$$

and

$$\tan\varphi = \frac{c\omega}{s-m\omega^2}$$

The vectors as shown in the diagram are fixed relative to one another and rotate with angular velocity ω.

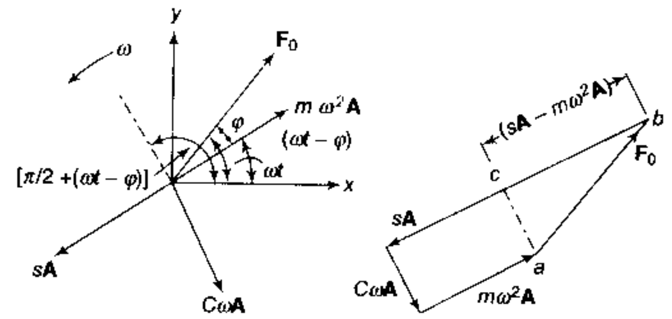


Fig. 18.21

18.12 MAGNIFICATION FACTOR

The ratio of the amplitude of the steady-state response to the static deflection under the action of force F_o is known as the *magnification factor* (*MF*).

$$MF = \frac{F_0 / \sqrt{(s-\omega^2)^2 + (c\omega)^2}}{F_0 / s}$$

$$= \frac{s}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{1}{\sqrt{\left(1 - \frac{m}{s}\omega^2\right)^2 + \left(\frac{c}{s}\omega\right)^2}}$$

$$= \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \qquad (18.41)$$

Thus, the magnification factor depends upon

(a) the ratio of frequencies, $\frac{\omega}{\omega_n}$, and

(b) the damping factor.

The plot of magnification factor against the ratio of frequencies (ω/ω_n) for different values of ζ is shown in Fig. 18.22(a). The curves show that as the damping increases or ζ increases, the maximum value of the magnification factor decreases and vice-versa. When there is no damping ($\zeta = 0$), it reaches infinity at $\omega/\omega_n = 1$, i.e., when the frequency of the forced vibrations is equal to the frequency of the free vibration. This condition is known as *resonance*.

In practice, the magnification factor cannot reach infinity owing to friction which tends to dampen the vibration. However, the amplitude can reach very high values.

Figure 18.22 (b) shows the plots of phase angle vs. frequency ratio (ω/ω_n) for different values of ζ. Observe the following:

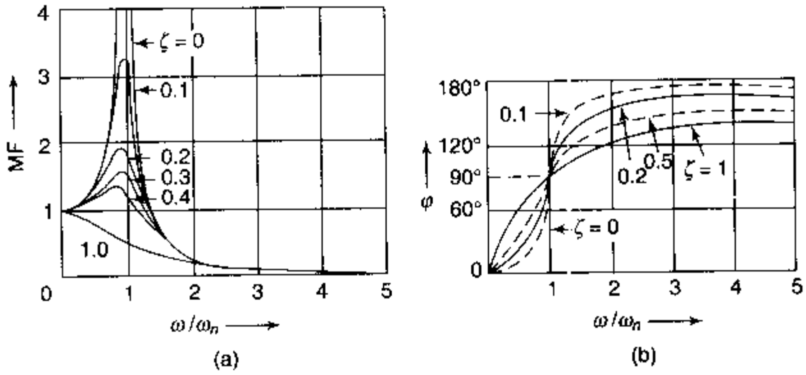


Fig. 18.22

- Irrespective of the amount of damping, the maximum amplitude of vibration occurs before the ratio ω/ω_n reaches unity or when the frequency of the forced vibration is less than that of the undamped vibrations.
- Phase angle varies from zero at low frequencies to 180° at very high frequencies. It changes very rapidly near the resonance and is 90° at resonance irrespective of damping.
- In the absence of any damping, phase angle suddenly changes from zero to 180° at resonance.

Example 18.11 *A machine part having a mass of 2.5 kg vibrates in a viscous medium. A harmonic exciting force of 30 N acts on the part and causes a resonant amplitude of 14 mm with a period of 0.22 second. Find the damping coefficient.*

If the frequency of the exciting force is changed to 4 Hz, determine the increase in the amplitude of the forced vibrations upon the removal of the damper.

Solution

$m = 2.5$ kg $\qquad F_o = 30$ N

$A = 14$ mm $\qquad T = 0.22$ s

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.22} = 28.56 \text{ rad/s}$$

(i) At resonance, $\omega = \omega_m$

or

$$\omega_n = \sqrt{\frac{s}{m}} = 28.56 \text{ rad/s}$$

or

$$\sqrt{\frac{s}{2.5}} = 28.56$$

$s = 2039$ N/m or 2.039 N/mm

Now,

$$A = \frac{F_0/s}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

or

$$A = \frac{F_0/s}{2\zeta} \qquad \left(\frac{\omega}{\omega_n} = 1\right)$$

or

$$0.014 = \frac{30/2039}{2\zeta}$$

or

$\zeta = 0.526$

$c = 2m\,\omega_n\,\zeta = 2 \times 2.5 \times 28.56 \times 0.526$

$= 75.04$ N/m/s

$= \underline{0.075\ 04 \text{ N/mm/s}}$

(ii) $\omega = f_n \times 2\pi = 4 \times 2\pi = 25.13$ rad/s

With damper

$$A = \frac{30/2039}{\sqrt{\left[1-\left(\frac{25.13}{28.56}\right)^2\right]^2 + \left[2 \times 0.526 \times \frac{25.13}{28.56}\right]^2}}$$

$$= \frac{30/2039}{\sqrt{(0.2258)^2 + (0.9248)^2}} = 0.0155 \text{ m}$$

Without damper: $\zeta = 0$

$$A = \frac{30/2039}{0.2258} = 0.0652 \text{ m}$$

$\therefore$ Increase in magnitude = 0.0652 − 0.0155 = 0.0497 m or 49.7 mm

Example 18.12 *A single-cylinder vertical diesel engine has a mass of 400 kg and is mounted on a steel chassis frame. The static deflection owing to the weight of the chassis is 2.4 mm. The reciprocating masses of the engine amounts to 18 kg and the stroke of the engine is 160 mm. A dashpot with a damping coefficient of 2 N/mm/s is also used to dampen the vibrations. In the steady-state of the vibrations, determine the*

(i) amplitude of the vibrations if the driving shaft rotates at 500 rpm

(ii) speed of the driving shaft when the resonance occurs

Solution

$m = 400$ kg $N = 500$ rpm

$c = 2000$ N/m/s $\Delta = 2.4$ mm

$r = 80$ mm $= 0.0024$ m

$$\omega = \frac{2\pi \times 500}{60} = 52.36 \text{ rad/s}$$

Now $s \times \Delta = mg$

$\therefore$ $s \times 0.0024 = 400 \times 9.81$

$s = 1.635 \times 10^6$ N/m

Centrifugal force due to reciprocating parts (or the static force),

$$F_0 = mr\omega^2 = 18 \times 0.08 \times (52.36)^2 = 3948 \text{ N}$$

(i) $$A = \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$ [Eq. (18.38)]

$$= \frac{3948}{\sqrt{[1.635 \times 10^6 - 400(52.36)^2]^2 + (2000 \times 52.36)^2}}$$

= 0.0072 m or 7.2 mm

(ii) *Resonant speed*

$$\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{1.635 \times 10^6}{400}} = 63.93 \text{ rad/s}$$

or $$\frac{2\pi N}{60} = 63.93$$

N = 610.5 rpm

Example 18.13 *A body having a mass of 15 kg is suspended from a spring which deflects 12 mm under the weight of the mass. Determine the frequency of the free vibrations. What is the viscous damping force needed to make the motion aperiodic at a speed of 1 mm/s?*

If, when damped to this extent, a disturbing force having a maximum value of 100 N and vibrating at 6 Hz is made to act on the body, determine the amplitude of the ultimate motion.

Solution

$m = 15$ kg $\Delta = 12$ mm

$F_0 = 100$ N $f = 6$ Hz

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.012}} = 4.55 \text{ Hz}$$

The motion becomes aperiodic when the damped frequency is zero or when it is critically damped ($\zeta = 1$) and

$$\omega = \omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.81}{0.012}} = 28.59 \text{ rad/s}$$

$c = c_c = 2m\,\omega_n = 2 \times 15 \times 28.59 = 857$ N/m/s = 0.857 N/mm/s

Thus, the force needed is 0.857 N at a speed of 1 mm/s.

$$A = \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

But $\omega = 2\pi \times f = 2\pi \times 6 = 37.7$ rad/s

and s can be found from

$$f_n = \frac{1}{2\pi}\sqrt{s/m}$$

or $$4.55 = \frac{1}{2\pi}\sqrt{s/15}$$

or $s = 12\,260$ N/m

$$\therefore A = \frac{100}{\sqrt{[12\,260 - 15 \times (37.7)^2]^2 + (857 \times 37.7)^2}}$$

= 0.002 98 m = 2.98 mm

18.12 VIBRATION ISOLATION AND TRANSMISSIBILITY

Vibrations are produced in machines having unbalanced masses. These vibrations will be transmitted to the foundation upon which the machines are installed. This is usually undesirable. To diminish the transmitted forces, machines are usually mounted on springs or dampers, or on some other vibration isolating material. Then the vibratory forces can reach the foundation only through these springs, dampers, or the isolating material used.

Transmissibility is defined as the ratio of the force transmitted (to the foundation) to the force applied. It is a measure of the effectiveness of the vibration isolating material.

As the transmitted force is the vector sum of the spring force (sA) and the damping force ($c\omega A$) which are at perpendicular to each other (Fig. 18.21),

$$F_t = \sqrt{(sA)^2 + (c\omega A)^2}$$

$$= A\sqrt{(s)^2 + (c\omega)^2}$$

$$= \frac{F_0}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}\sqrt{s^2 + (c\omega)^2}$$

$$= \frac{F_0\sqrt{1 + \left(\frac{c}{s}\omega\right)^2}}{\sqrt{\left(1 - \frac{m}{s}\omega^2\right)^2 + \left(\frac{c}{s}\omega\right)^2}}$$

$$= \frac{F_0\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$$

Transmissibility,

$$\varepsilon = \frac{F_t}{F_0} = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \qquad (18.42)$$

At resonance,

$$\frac{\omega}{\omega_n} = 1,$$

∴

$$\varepsilon = \frac{\sqrt{1 + (2\zeta)^2}}{2\zeta} \qquad (18.43)$$

when no damper is used, $\zeta = 0$
and

$$\varepsilon = \frac{1}{\pm[1 - (\omega/\omega_n)^2]} \qquad (18.44)$$

Transmissibility has been plotted against ω/ω_n for different values of ζ in Fig.18.23. Note that

(i) when $\omega/\omega_n < \sqrt{2}$, ε is more than 1, i.e., the transmitted force is always more than the exciting force

(ii) when $\omega/\omega_n > \sqrt{2}$, ε is less than 1, i.e., the transmitted force is always less than the exciting force

(iii) when $\omega/\omega_n = \sqrt{2}$, ε is 1, i.e., the transmitted force is equal to the exciting force

(iv) when $\omega/\omega_n > 1$, the transmitted force is infinite; if damping is used, the magnitude of the transmitted force can be reduced

(v) when $\omega/\omega_n = \sqrt{2}$, ε increases as the damping is increased

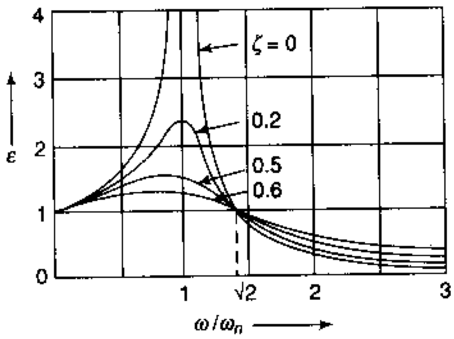


Fig. 18.2[illegible]

Thus in a system where ω/ω_n can vary from zero to higher values, dampers should not be used. Instead, stops may be provided to limit the resonance amplitude (at resonance, the amplification factor is infinitely).

Example 18.14 ***A refrigerator unit having a mass of 35 kg is to be supported on three springs, each having a spring stiffness s. The unit operates at 480 rpm. Find the value of stiffness s if only 10% of the shaking force is allowed to be transmitted to the supporting structure.***

Solution As no damper is used,

$$\varepsilon = \frac{1}{\pm\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$\omega = \frac{2\pi \times 480}{60} = 16\pi \quad \text{and} \quad \varepsilon = 0.1$$

$$\therefore \quad 0.1 = \frac{1}{\pm\left[1-\left(\frac{16\pi}{\omega_n}\right)^2\right]}$$

$$\pm\left[0.1 - 0.1\left(\frac{16\pi}{\omega_n}\right)^2\right] = 1$$

If the positive sign is taken, $\frac{16\pi}{\omega_n} = \sqrt{-9}$ which is not possible.

Therefore taking the negative sign, $\frac{16\pi}{\omega_n} = \sqrt{11}$

or $\quad \omega_n = 15.15$ rad/s

or $\quad \sqrt{\frac{s}{m}} = \sqrt{\frac{s}{35}} = 15.15$

Equivalent stiffness,

$s = 8037$ N/m $= 8.037$ N/mm

Stiffness of each spring $= \frac{8.037}{3} = \underline{2.679 \text{ N/mm}}$

Example 18.15 ***A machine supported symmetrically on four springs has a mass of 80 kg. The mass of the reciprocating parts is 2.2 kg which move through a vertical stroke of 100 mm with simple harmonic motion. Neglecting damping, determine the combined stiffness of the springs so that the force transmitted to the foundation is 1/20th of the impressed force. The machine crankshaft rotates at 800 rpm.***

If, under actual working conditions, the damping reduces the amplitudes of successive vibrations by 30%, find the

(i) force transmitted to the foundation at 800 rpm

(ii) force transmitted to the foundation at resonance

(iii) amplitude of the vibrations at resonance

Solution

$M = 80$ kg $\qquad \varepsilon = \frac{1}{20} = 0.05$

$m = 2.2$ kg $\qquad N = 800$ rpm

$r = \frac{100}{2} = 50$ mm

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 800}{60}$$
$$= 83.78 \text{ rad/s}$$

In the absence of damping,

$$\varepsilon = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$

or

$$0.05 = \frac{1}{\left(\frac{83.78}{\omega_n}\right)^2 - 1}$$

$$\omega_n = 18.28 \text{ rad/s}$$

or

$$\sqrt{\frac{s}{M}} = \sqrt{\frac{s}{80}} = 18.28$$

∴ combined stiffness, $s = 26\ 739$ N/m

$= 26.739$ N/mm

(i) $$\frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{1}{1-0.3}\right)$$

$$\frac{\zeta^2}{1-\zeta^2} = 0.003\ 23$$

$$\zeta = 0.0567$$

$$\varepsilon = \frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$$

$$= \frac{\sqrt{1+\left(2\times 0.0567 \times \frac{83.78}{18.28}\right)^2}}{\sqrt{\left[1-\left(\frac{83.78}{18.28}\right)^2\right]^2 + \left(2\times 0.0567\times\frac{83.78}{18.28}\right)^2}}$$

$= 0.0563$

The maximum unbalanced force on the machine due to the reciprocating parts,

$F = mr\omega^2 = 2.2 \times 0.05 \times (83.78)^2 = 772.1$ N

$$\varepsilon = \frac{F_t}{F}$$

or

$$0.0563 = \frac{F_t}{772.1}$$

or

$F_t = 43.47$ N

(ii) At resonance, $\frac{\omega}{\omega_n} = 1$

$$\varepsilon = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta}$$

$$= \frac{\sqrt{1+(2\times 0.0567)^2}}{2\times 0.0567} = 8.875$$

Maximum unbalanced force on the machine due to reciprocating parts at resonance, i.e., when $\omega = \omega_n$,

$F = 2.2 \times 0.05 \times (18.28)^2 = 36.76$ N

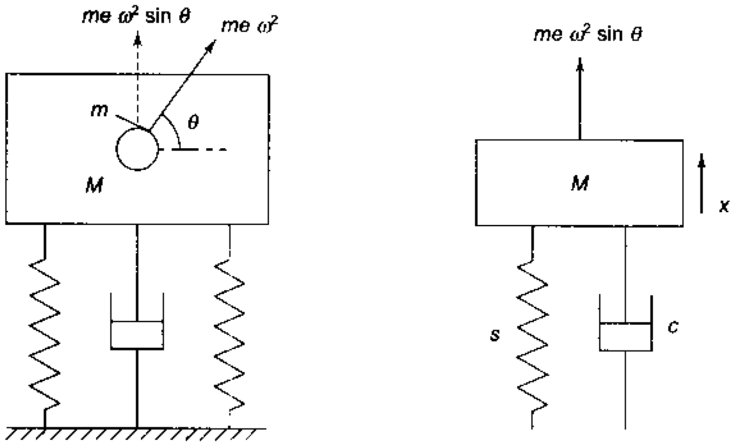
$F_t = \varepsilon \times F = 8.875 \times 36.76 = 326.25$ N

(iii) $$\text{Amplitude} = \frac{\text{Force transmitted at resonance}}{\text{Stiffness}}$$

$$= \frac{326.25}{26.739} = \underline{12.2 \text{ mm}}$$

18.14 FORCING DUE TO UNBALANCE

All types of rotating machinery such as electric motor, turbine or a pump always consist of some amount of unbalance left in them even though they are carefully balanced on balancing machines. The net unbalance in such machines may be represented by a mass m rotating with its centre of mass at a distance e from axis of rotation (Fig. 18.24). If M is the total or the vibrating mass of the machine including the unbalanced mass m,



Vibrating system

Equivalent system

Fig. 18.24

the centrifugal force acting outwards from the centre of rotation = $me\omega^2$

Assume that the system is constrained to move vertically. The equation of motion in the vertical direction can be written as

$$m\ddot{x} + c\dot{x} + sx = me\omega^2 \sin\omega t \qquad (18.45)$$

The equation is similar to Eq. 18.36 except that F_o is replaced by $me\omega^2$, assuming that ω is constant, the force represented by $me\omega^2$ is constant. Thus the steady state solution for the equation can be written directly, i.e.,

$$x = \frac{me\omega^2}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \varphi) \qquad (18.46)$$

The amplitude,

$$A = \frac{me\omega^2 / s}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}}$$

But

$$me\frac{\omega^2}{s} = me\frac{\omega^2 / M}{s / M} = \frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2$$

Therefore,

$$\frac{A}{\frac{me}{M}} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \qquad (18.47)$$

The equation provides the steady-state amplitude as a function of damping factor and frequency ratio. This has been plotted in Fig. 18.25. It shows that at higher values of frequency ratio ω / ω_n, the amplitude can be reduced by mass and eccentricity of the rotating unbalance. The equation for phase angle remains the same as Eq. 18.40.

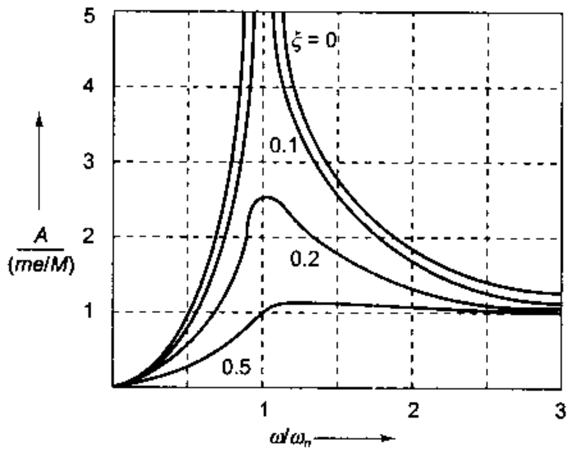


Fig. 18.2

The above analysis can easily be extended to the case of a reciprocating unbalance (Fig. 18.26). The inertia force due to reciprocating mass is approximately equal to

$$= mr\omega^2\left(\cos\theta + \frac{\cos 2\theta}{l/r}\right) \qquad \text{(Eq. 13.18)}$$

If l/r ratio is large, the second harmonic may be neglected, and the equation of motion may be written as

$$m\ddot{x} + c\dot{x} + sx = mr\omega^2 \cos\omega t$$

which is similar to that for rotating unbalance and can easily be analysed.

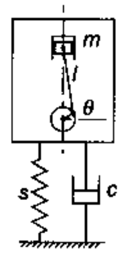


Fig. 18.2

FORCING DUE TO SUPPORT MOTION

In case of vehicles, the excitation of the system is through the support or base instead of directly to the mass. Assuming that the support is excited by a harmonic motion (Fig. 18.27),

$$y = Y \sin\omega t \qquad (18.48)$$

and the displacement of mass x is more as compared to the displacement of y in the considered position.

The equation of motion can be written as

$$m\ddot{x} + c(\dot{x} - \dot{y}) + s(x - y) = 0$$

or $$m\ddot{x} + c\dot{x} + sx = c\dot{y} + sy$$

$$= c\,Y\omega\cos\omega t + s\,Y\sin\omega t \qquad (\dot{y} = Y\omega\cos\omega t)$$

$$= Y[c\omega\cos\omega t + s\sin\omega t]$$

Let $c\omega = K\sin\alpha$ and $s = K\sin\alpha$

So that $$K = \sqrt{(c\omega)^2 + s^2} \quad \text{and} \quad \alpha = \tan^{-1}\frac{c\omega}{s} = \tan^{-1}\left(2\zeta\frac{\omega}{\omega_n}\right)$$

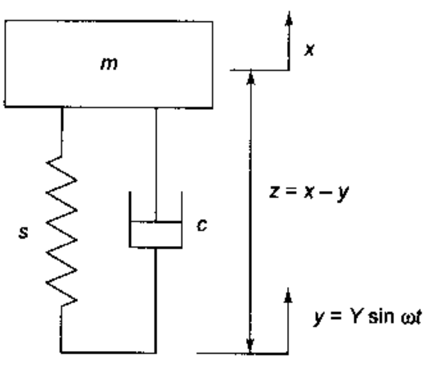


Fig. 18.2

Thus the equation transforms to

$$m\ddot{x} + c\dot{x} + sx = Y[c\omega\cos\omega t + s\sin\omega t]$$
$$= Y[K\sin\alpha\ \cos\omega t + K\sin\alpha\sin\omega t]$$
$$= YK\sin(\omega t + \alpha)$$

The steady state solution is similar to that of 18.38,

$$x = \frac{YK}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}\sin(\omega t + \alpha - \varphi) \quad (18.49)$$

The amplitude is

$$A = \frac{YK}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

$$\frac{A}{Y} = \frac{\sqrt{s + (c\omega)^2}}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

which can be transformed into dimensionless form,

$$\frac{A}{Y} = \frac{\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad (18.50)$$

ϕ is given by

$$\varphi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Comparing Eqs (18.48) and (18.49), it is observed that the motion of mass leads the support motion through an angle $(\alpha - \phi)$ or lags by angle $(\phi - \alpha)$.

$$\phi - \alpha = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} - \tan^{-1} 2\zeta\frac{\omega}{\omega_n} \quad (18.51)$$

- From the Eq. 18.50 it can be noted that in case the exciting frequency ω is very small as compared to ω_n or ω is negligible, the ratio A/Y approaches one or the complete system vibrates as a rigid body.
- If $\omega >> \omega_n$, ω/ω_n approaches infinity and thus, A/Y approaches zero or the body is stationary.

The ratio A/Y is usually known as *displacement* or *amplitude transmissibility*. The plots of transmissibility and phase lag are similar to those for force transmissibility given in Figs 18.23 and 18.22(b).

Relative Amplitude Let z be the displacement of the of the mass relative to the support so that

$z = x - y$

or $x = y + z$

As $y = Y\sin\omega t$

$\therefore \quad \dot{y} = Y\omega\cos\omega t \quad \text{and} \quad \ddot{y} = -Y\omega^2\sin\omega t$

The equation of motion can be written as

$$m(\ddot{y} + \ddot{z}) + c\dot{z} + sz = 0$$

or

$$m\ddot{z} + c\dot{x} + sx = -m\ddot{y} = mY\omega^2 \sin \omega t$$

The equation is similar to Eq. (in sec 18.45). Thus the steady state response is

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \qquad (18.52)$$

where Z is the steady state relative amplitude. The equation for phase angle remains same as Eq. 18.40.

SECTION II (TRANSVERSE VIBRATIONS)

Natural vibrations of shafts and beams under different types of loads and end conditions have been explained in the following sections:

18.16 SINGLE CONCENTRATED LOAD

In case of shafts and beams of negligible mass carrying a concentrated mass, the force is proportional to the deflection of the mass from the equilibrium position and the relation derived for natural frequency of longitudinal vibrations holds good, i.e.,

$$f_n = \frac{1}{2\pi}\sqrt{g / \Delta}$$

where $\Delta = \dfrac{mgl^3}{3EI}$ for cantilevers, supporting a concentrated mass at the free end

$$= \frac{mga^2b^2}{3EIl} \text{ for simply supported beams}$$

$$= \frac{mga^3b^3}{3EIl^3} \text{ for beams fixed at both ends}$$

These cases have been shown in Fig. 18.28.

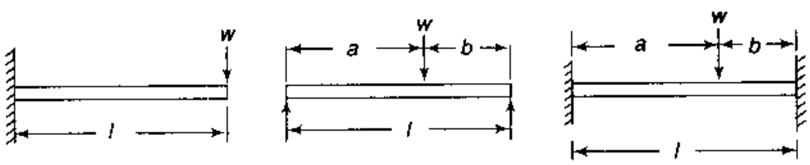


Fig. 18.28

A shaft supported in long bearings is assumed to have both ends fixed while one in short bearings is considered to be simply supported.

Example 18.16 *A shaft supported freely at the ends has a mass of 120 kg placed 250 mm from one end. The shaft diameter is 40 mm. Determine the frequency of the natural transverse vibrations if the length of the shaft is 700 mm, E = 200 GN/m².*

Solution

$m = 120$ kg $E = 200 \times 10^9$ N/m²
$l = 0.7$ m $d = 0.04$ m
$a = 0.25$ m $b = 0.7 - 0.25 = 0.45$ m

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} \times (0.04)^4 = 0.1256 \times 10^{-6}\,\text{m}^4$$

$$\Delta = \frac{mga^2b^2}{3EIl}$$

$$= \frac{120 \times 9.81 \times (0.25)^2 \times (0.45)^2}{3 \times 200 \times 10^9 \times 0.1256 \times 10^{-6} \times 0.7}$$

$$= 0.282 \times 10^{-3}\,\text{m}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.282 \times 10^{-3}}} = \underline{29.68\text{ Hz}}$$

18.17 UNIFORMLY LOADED SHAFT

Figure 18.29 shows a shaft supported at its ends and carrying a uniform mass.

Let m = distributed mass per unit length
l = length of the shaft

The shaft makes transverse vibrations due to elastic forces. At any instant, let it be deflected by an amount y at a distance x from the end A. The vibrations being free and due to elastic forces, will be of simple-harmonic-motion type.

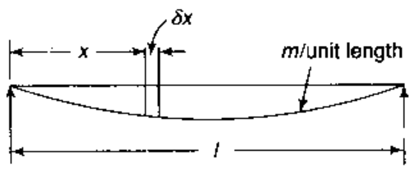


Fig. 18.29

From the theory of bending of shafts,

$$EI\frac{d^4y}{dx^4} = \text{dynamic load per unit length}$$

$$= \text{centrifugal force per unit length.}$$

$$= my\,\omega^2$$

or $$\frac{d^4y}{dx^4} - \frac{my\omega^2}{EI} = 0 \quad (18.53)$$

or $$\frac{d^4y}{dx^4} - \lambda^4 y = 0$$

where $$\lambda^4 = \frac{m\omega^2}{EI}$$

The auxiliary equation is

$$(D^4 - \lambda^4)\,y = 0$$

This gives

$$D = \pm\,\lambda \text{ and } \pm\,i\lambda$$

The solution will be of the form

$$y = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x \quad \text{(i)}$$

This is the general expression for the deflection in case of uniformly loaded shafts. Constants A, B, C and D have to be found from the end conditions.

Simply Supported Shaft

The boundary conditions are

(a) $y = 0$ at $x = 0$ and l

(b) $\frac{d^2y}{dx^2} = 0$ at $x = 0$ and l (bending moment is zero at ends)

When $x = 0$, $y = 0$; $B + D = 0$ (ii)

When $x = l$, $y = 0$

$$A \sin \lambda l + B \cos \lambda l + C \sinh \lambda l + D \cosh \lambda l = 0 \quad \text{(iii)}$$

Differentiating (i) with respect to x twice,

$$\frac{dy}{dx} = \lambda (A \cos \lambda x - B \sin \lambda x + C \cosh \lambda x + D \sinh \lambda x)$$

$$\frac{d^2y}{dx^2} = \lambda^2(-A \sin \lambda x - B \cos \lambda x + C \sinh \lambda x + D \cosh \lambda x)$$

When $x = 0$,

$$\frac{d^2y}{dx^2} = 0$$

$\therefore$

$$\lambda^2 (-B + D) = 0 \quad \text{(iv)}$$

When $x = l$,

$$\frac{d^2y}{dx^2} = 0$$

$$\lambda^2(-A \sin \lambda l - B \cos \lambda l + C \sinh \lambda l + D \cosh \lambda l) = 0 \quad \text{(v)}$$

From (ii) and (iv)

$B = 0$ and $D = 0$

Thus (iii) and (v) can be written as

$$A \sin \lambda l + C \sinh \lambda l = 0$$

and

$$-A \sin \lambda l + C \sinh \lambda l = 0$$

Adding these, we get

$$C \sinh \lambda l = 0$$

Subtracting,

$$A \sin \lambda l = 0$$

$\sinh \lambda l$ cannot be zero, because if $\lambda = 0$, $\lambda^4 = 0$

or

$$\frac{m\omega^2}{EI} = 0$$

or $$\frac{m}{EI}(2\pi f_n)^2 = 0$$

or $$f_n = 0$$

which means that the system does not vibrate.

$\therefore$ $$C = 0$$

Thus Eq. (i) reduces to

$$y = A \sin \lambda x \qquad (B, C \text{ and } D \text{ are zero})$$

Now, when $A \sin \lambda l = 0$, A cannot be zero as B, C and D are already zero and if A is also zero, there are no vibrations.

$\therefore$ $$\sin \lambda l = 0$$

or

$$\lambda l = 0, \pi, 2\pi, 3\pi, \ldots$$

But λl cannot be equal to zero; if so, there will be no vibration.

$\therefore$ $$\lambda = \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, \ldots$$

or

or $$\left[\frac{m\omega^2}{EI}\right]^{1/4} = \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, \ldots$$

or $$\omega^{1/2} = \frac{\pi}{l}\left(\frac{EI}{m}\right)^{1/4}, \frac{2\pi}{l}\left(\frac{EI}{m}\right)^{1/4}, \frac{3\pi}{l}\left(\frac{EI}{m}\right)^{1/4}, \ldots$$

$$\omega = (2\pi f_n) = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{m}}, \frac{4\pi^2}{l^2}\sqrt{\frac{EI}{m}}, \frac{9\pi^2}{l^2}\sqrt{\frac{EI}{m}}, \ldots$$

$$f_n = \frac{\pi}{2}\sqrt{\frac{EI}{ml^4}}, \frac{4\pi}{2}\sqrt{\frac{EI}{ml^4}}, \frac{9\pi}{2}\sqrt{\frac{EI}{ml^4}}$$

A simply supported shaft carrying a uniformly distributed mass has maximum deflection at the mid-span.

$$\Delta = \frac{5mgl^4}{384EI}$$

or

$$\frac{EI}{ml^4} = \frac{5g}{384\Delta}$$

Then, taking the smallest value of f_n,

$$f_n = \frac{\pi}{2}\sqrt{\frac{5g}{384\Delta}} \qquad (18.54)$$

This is the lowest frequency of transverse vibrations and is called the fundamental frequency.

As the equation for the displacement is $y = A \sin \lambda l$, and at node points, $y = 0$

$\therefore \qquad 0 = A \sin \lambda x = A \sin \frac{\pi}{l} x$

or $\qquad \frac{\pi}{l} x = 0$, i.e., $x = 0$ and l

This means a node at each end.

The next higher frequency is four times the fundamental frequency.

$$0 = A \sin \lambda x = A \sin \frac{2\pi}{l} x$$

or $\qquad \frac{2\pi}{l} x = 0 \quad$ i.e. $x = 0$, $l/2$ and l

i.e., it has three nodes, two at the ends and one at the centre.

The next higher frequency is nine times the fundamental frequency. It has four nodes dividing the shaft into three equal parts, and so on (Fig. 18.30).

Thus a simply supported shaft will have an infinite number of frequencies under a uniformly distributed load.

Similarly, the cases of cantilevers and shafts fixed at both ends can be considered. The end conditions will be as follows.

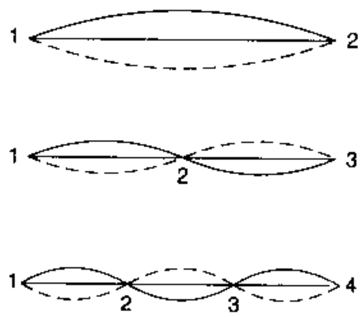


Fig. 18.30

(i) Cantilevers

$y = 0$ at $x = 0$ (zero deflection)

$\frac{dy}{dx} = 0$ at $x = 0$ (zero slope)

$\frac{d^2 y}{dx^2} = 0$ at $x = l$ (zero bending moment)

$\frac{d^3 y}{dx^3} = 0$ at $x = l$ (zero shear force)

and $\qquad \Delta = \frac{mgl^3}{8EI}$

(ii) Both Ends Fixed

$y = 0$ at $x = 0$ and l

$\frac{dy}{dx} = 0$ at $x = 0$ and l

and $\qquad \Delta = \frac{mgl^4}{384EI}$

18.18 SHAFT CARRYING SEVERAL LOADS

There are two methods to find the natural frequency of the system:

(i) Dunkerley's method which is semi-empirical. This gives approximate results but is simple.
(ii) The energy method which gives accurate results but involves heavy calculations if there are many loads.

(i) Dunkerley's Method

Let W_1, W_2, W_3,.... be the concentrated loads on the shaft due to masses m_1, m_2, m_3, ... and Δ_1, Δ_2, Δ_3, ... the static deflections of this shaft under each load when that load acts alone on the shaft. Let the shaft carry a uniformly distributed mass of m per unit length over its whole span and the static deflection at mid-span due to the load of this mass be Δ_s. Also, let

f_n = frequency of transverse vibration of the whole system

f_{ns} = frequency with the distributed load acting along

f_{n1}, f_{n2}, f_{n3}, ... = frequency of transverse vibrations when each of W_1, W_2, W_3,... acts alone.

Then, according do Dunkerley's empirical formula,

$$\frac{1}{f_n^2} = \frac{1}{f_{n1}^2} + \frac{1}{f_{n2}^2} + \frac{1}{f_{n3}^2} + \ldots\ldots + \frac{1}{f_{ns}^2} \qquad (18.55)$$

where

$$f_{n1} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_1}} = \frac{\sqrt{9.81}}{2\pi}\frac{1}{\sqrt{\Delta_1}} = \frac{0.4985}{\sqrt{\Delta_1}}$$

Similarly,

$$f_{n2} = \frac{0.4985}{\sqrt{\Delta_2}};\ f_{n3} = \frac{0.4985}{\sqrt{\Delta_3}}, \text{ and so on.}$$

$$f_{ns} = \frac{\pi}{2}\sqrt{\frac{5g}{384\Delta_s}} = \frac{\pi}{2}\sqrt{\frac{5 \times 9.81}{384}} \times \frac{1}{\sqrt{\Delta_s}} = \frac{0.5614}{\sqrt{\Delta_s}}$$

$$\frac{1}{f_n^2} = \frac{1}{(0.4985)^2}(\Delta_1 + \Delta_2 + \Delta_3 + \ldots) + \frac{1}{(0.5614)^2}\Delta_s$$

$$= \frac{1}{(0.4985)^2}\left(\Delta_1 + \Delta_2 + \Delta_3 + \ldots + \frac{\Delta_s}{1.27}\right)$$

$$f_n = \frac{0.4985}{\sqrt{\Delta_1 + \Delta_2 + \Delta_3 + \ldots + \frac{\Delta_s}{1.27}}} \qquad (18.56)$$

(ii) Energy Method

Consider a shaft with negligible mass, carrying point loads W_1, W_2, W_3,... due to masses m_1, m_2, m_3,... as shown in Fig. 18.31. Let y_1, y_2, y_3,... be the total deflection these loads.

In the extreme positions of the shaft, it possesses maximum potential energy and no kinetic energy, whereas in the mean position, it possesses maximum kinetic energy and no potential energy. Thus, the maximum potential energy of the shaft can be made equal to its maximum kinetic energy.

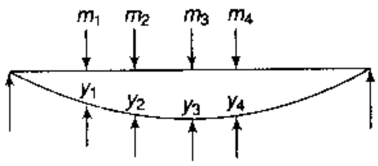


Fig. 18.31

$$\text{Maximum } PE = \frac{1}{2}W_1y_1 + \frac{1}{2}W_2y_2 + \frac{1}{2}W_3y_3 + ...$$
$$= \frac{g}{2}(m_1y_1 + m_2y_2 + m_3y_3 + ...)$$
$$= \frac{g}{2}\Sigma my$$
$$\text{Maximum } KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + ...$$
$$= \frac{1}{2}m_1(\omega y_1)^2 + \frac{1}{2}m_2(\omega y_2)^2 + \frac{1}{2}m_3(\omega y_3)^2 + ...$$
$$= \frac{\omega^2}{2}(m_1y_1^2 + m_2y_2^2 + m_3y_3^3 + ...)$$
$$= \frac{\omega^2}{2}\Sigma my^2$$

where ω is the circular frequency of vibration. Equating maximum PE and maximum KE,

$$\frac{g}{2}\Sigma my = \frac{\omega^2}{2}\Sigma my^2$$
$$\omega = \sqrt{\frac{g\,\Sigma my}{\Sigma my^2}}$$
$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g\,\Sigma my}{\Sigma my^2}} \qquad (18.57)$$

Example 18.17 *A shaft of 40-mm diameter and 2.5-m length has a mass of 15 kg per metre length. It is simply supported at the ends and carries three masses of 90 kg, 140 kg and 60 kg at 0.8 m, 1.5 m and 2 m respectively from the left support. Taking E = 200 GN/m², find the frequency of the transverse vibrations.*

Solution

$d = 40 \text{ mm} = 0.04 \text{ m} \qquad l = 2.5 \text{ m}$

$$I = \frac{\pi}{64}\times d^4 = \frac{\pi}{64}\times(0.04)^4$$
$$= 0.1257\times10^{-6}\ \text{m}^4$$

We have,

$$f_n = \frac{0.4985}{\sqrt{\Delta_1 + \Delta_2 + \Delta_3 + ... + \frac{\Delta_s}{1.27}}}$$

$$\Delta_1 = \frac{mga^2b^2}{3EIl}$$

Here $m = 90$ kg, $a = 0.8$ m and $b = 1.7$ m.

$$\therefore \quad \Delta_1 = \frac{90\times9.81\times(0.8)^2\times(1.7)^2}{3\times200\times10^9\times0.1257\times10^{-6}\times2.5}$$
$$= 0.008\ 66 \text{ m}$$

For Δ_2, $m = 140$ kg, $a = 1.5$ m, $b = 1$ m

$$\therefore \quad \Delta_1 = \frac{140\times9.81\times(1.5)^2\times(1)^2}{3\times200\times10^9\times0.1257\times10^{-6}\times2.5}$$
$$= 0.1639 \text{ m}$$

For Δ_3, $m = 60$ kg, $a = 2$ m, $b = 0.5$ m

$$\therefore \quad \Delta_3 = \frac{60\times9.81\times(2)^2\times(0.5)^2}{3\times200\times10^9\times0.1257\times10^{-6}\times2.5}$$
$$= 0.003\ 12 \text{ m}$$

$$\Delta_s = \frac{5mgl^4}{384EI} = \frac{5 \times 15 \times 9.81 \times (2.5)^4}{384 \times 200 \times 10^9 \times 0.1257 \times 10^{-6}}$$
$$= 0.002\ 98 \text{ m}$$

$$f_n = \frac{0.4985}{\sqrt{0.008\,66 + 0.016\,39 + 0.003\,12 + \dfrac{0.002\,98}{1.27}}}$$
$$= 2.85 \text{ Hz}$$

18.19 WHIRLING OF SHAFTS

When a rotor is mounted on a shaft, its centre of mass does not usually coincide with the centre line of the shaft. Therefore, when the shaft rotates, it is subjected to a centrifugal force which makes the shaft bend in the direction of eccentricity of the centre of mass. This further increases the eccentricity, and hence the centrigugal force. In this way, the effect is cumulative and ultimately the shaft may even fail. The bending of the shaft depends upon the eccentricity of the centre of mass of the rotor as also upon the speed at which the shaft rotates.

Critical or *whirling* or *whipping speed* is the speed at which the shaft tends to vibrate violently in the transverse direction.

It has been observed that if the critical speed is instantly run through, the shaft again becomes almost straight. But at some other speed, the same phenomenon recurs, the only difference being that the shaft now bends in two bows, and so on.

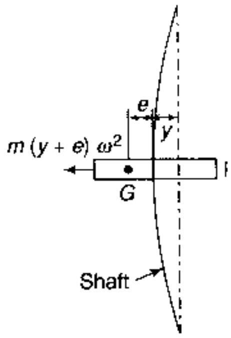


Fig. 18.32

Figure 18.32 shows a rotor having a mass m attached to a shaft.

Let s = stiffness of shaft

e = initial eccentricity of centre of mass of rotor

m = mass of rotor

y = additional deflection of rotor due to centrifugal force

ω = angular velocity of shaft.

Then

Centrifugal force $= m(y + e)\,\omega^2$

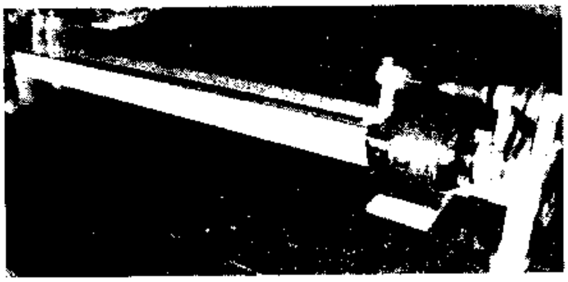
Force resisting the deflection $= sy$

For equilibrium,

$$sy = m(y + e)\,\omega^2 = my\,\omega^2 + me\,\omega^2$$

or $\quad y(s - m\,\omega^2) = me\,\omega^2$

$$y = \frac{me\omega^2}{s - m\omega^2} = \frac{e}{\dfrac{s}{m\omega^2} - 1} = \frac{e}{\left(\dfrac{\omega_n}{\omega}\right)^2 - 1}$$



Lab set up to find the whirling speed of shafts

Thus when $\omega = \omega_n$, the deflection y is infinitely large (resonance occurs) and the speed ω is the critical speed, i.e.

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\Delta}} \qquad (18.58)$$

If the speed of the shaft is increased rapidly beyond the critical speed, $\omega > \omega_n$ or $(\omega_n/\omega)^2 < 1$ or y is negative. This means that the shaft deflects in the opposite direction. As the speed continues to increase, y approaches the value $-e$ or the centre of mass of the rotor approaches the centre line of rotation. This principle is used in running high-speed turbines by speeding up the rotor rapidly or beyond the critical speed. When y approaches the value of $-e$, the rotor runs steadily.

Example 18.18 *Determine the critical speed of the shaft of Example 18.17 loaded in the same way.*

Solution The critical speed of the shaft in revolutions per second is equal to the natural frequency of transverse vibrations in Hz.

$f_n = 2.85$ Hz

$N_c = 2.85$ rps $= (2.85 \times 60)$ rpm $= 171$ rpm

Example 18.19 *A rotor has a mass of 12 kg and is mounted midway on a 24-mmdiameterhorizontalshaft supported at the ends by two bearings. The bearings are 1 m apart. The shaft rotates at 2400 rpm. If the centre of mass of the rotor is 0.11 mm away from the geometric centre of the rotor due of a certain manufacturing defect, find the amplitude of the steady-state vibration and the dynamic force transmitted to the bearing. E = 200 GN/m².*

Solution Assuming the bearings to be short so that the shaft can be assumed to be simply supported,

$m = 12$ kg $\qquad l = 1$ m

$d = 0.024$ m $\qquad N = 2400$ rpm

$e = 0.11$ mm $\qquad E = 200 \times 10^9$ N/m²

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} \times (0.024)^4 = 16.3 \times 10^{-9} \text{ m}^4$$

$$\Delta = \frac{mgl^3}{48EI} = \frac{12 \times 9.81 \times (1)^3}{48 \times 200 \times 10^9 \times 16.3 \times 10^{-9}}$$

$= 0.000\ 752$ m

$$\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.81}{0.000\ 752}} = 114.2 \text{ rad/s}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60} = 251.3 \text{ rad/s}$$

Amplitude,

$$y = \frac{e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{0.11}{\left(\frac{114.2}{251.3}\right)^2 - 1} = -0.139 \text{ mm}$$

$= -0.000\ 139$ m

The negative sign indicates that the displacement is out of phase with the centrifugal force.

Dynamic force on the bearings $= sy$

$= m\omega_n^2 y$

$= 12 \times (114.2)^2 \times 0.000\ 139$

$= \underline{21.7 \text{ N}}$

The total load on each bearing can also be found.

Total load on each bearing

$$= \frac{mg}{2} + \frac{sy}{2} = \frac{12 \times 9.81}{2} + \frac{21.7}{2} = 69.7 \text{ N}$$

Example 18.20 *The following data relate to a shaft held in long bearings.*

Length of shaft	*= 1.2 m*
Diameter of shaft	*= 14 m*
Mass of a rotor at midpoint	*= 16 kg*
Eccentricity of centre of mass of rotor from centre of rotor	*= 0.4 mm*
Modulus of elasticity of shaft material	*= 200 GN/m²*
Permissible stress in shaft material	*= 70 × 10⁶ N/m²*

Determine the critical speed of the shaft and the range of speed over which it is unsafe to run the shaft. Assume the shaft to be massless.

Solution

$m = 16$ kg $\qquad e = 0.0004$ m

$l = 1.2$ m $\qquad E = 200 \times 10^9$ N/m²

$d = 0.014$ m $\qquad f = 70 \times 10^6$ N/m²

(i) As the shaft is held in long bearings, it may be assumed to be fixed at the ends.

$$\Delta = \frac{mgl^3}{192EI} = \frac{16 \times 9.81 \times (1.2)^3}{192 \times 200 \times 10^9 \times \frac{\pi}{64} \times (0.014)^4}$$

$= 0.003\,75$ m

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.003\,75}} = 8.143 \text{ Hz}$$

The critical speed of the shaft in rps is equal to the natural frequency of transverse vibrations in Hz, i.e.,

$N_c = 8.143$ rps = 489 rpm

(ii) When the shaft rotates, additional dynamic load on the shaft can be obtained from the relation

$$\frac{M}{I} = \frac{f}{y}$$

$$\frac{\frac{W_1 l}{8}}{\frac{\pi}{64} \times d^4} = \frac{f}{d/2}$$

or

$$\frac{\frac{W_1 \times 1.2}{8}}{\frac{\pi}{64} \times (0.014)^4} = \frac{70 \times 10^6}{\frac{0.014}{2}}$$

$W_1 = 125.7$ N

Additional deflection due to this load

$$= \frac{W_1}{W} \times \Delta$$

$$= \frac{W_1}{mg} \times \Delta$$

$$= \frac{125.7}{16 \times 9.81} \times 0.003\,75$$

$= 0.003$ m

Also,

$$\text{Additional deflection, } y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

$$0.003 = \frac{\pm 0.0004}{\left(\frac{N_c}{N}\right)^2 - 1}$$

or

$$\left(\frac{489}{N}\right)^2 - 1 = \pm 0.1333$$

N = 459 and 525

Thus, the range of unsafe speed is from 459 rpm to 525 rpm.

SECTION III (TORSIONAL VIBRATION)

18.20 FREE TORSIONAL VIBRATIONS (SINGLE ROTOR)

Consider a uniform shaft of length l rigidly fixed at its upper end and carrying a disc of moment of inertia I at its lower end (Fig. 18.33). The shaft is assumed to be massless. If the disc is given a twist about its vertical axis and then released, it will start oscillating about the axis and will perform torsional vibrations.

Let θ = angular displacement of the disc from its equilibrium position at any instant

q = torsional stiffness of the shaft

= torque required to twist the shaft per radian within elastic limits $= \left(\frac{GJ}{l}\right)$

where

G = modulus of rigidly of the shaft material

J = polar moment of inertia of the shaft cross-section

At any instant, the torques acting on the disc are

- Inertia torque $= -I\ddot{\theta}$
- Restoring torque (spring torque) $= -q\theta$

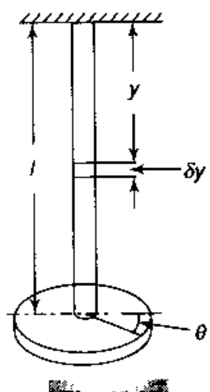


Fig. 18.33

Negative signs have been used as both of these torques act opposite to the angular displacement.

For equilibrium, the sum of all torques acting on the disc must be zero.

Therefore,

$$I\ddot{\theta} + q\theta = 0$$

or

$$\ddot{\theta} + \frac{q}{I}\theta = 0$$

This is the equation of simple harmonic motion.

$$\omega_n = \sqrt{\frac{q}{I}} \tag{18.59}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} \tag{18.59a}$$

18.21 INERTIA EFFECT OF MASS OF SHAFT

Let I_1 = moment of inertia of shaft

ω = angular velocity of free end

Consider an element of length δy at a distance y from the fixed end. Then

$$KE \text{ of element} = \frac{1}{2} \times (MOI \text{ of element}) \times (\text{angular velocity})^2$$

$$= \frac{1}{2} \times \left(I_1 \frac{\delta y}{l}\right)\left(\frac{\omega y}{l}\right)^2$$

$$\text{KE of shaft} = \int_0^l \frac{1}{2} \times \frac{I_1}{l}\left(\frac{y}{l}\omega\right)^2 dy$$

$$= \frac{I_1\omega^2}{2l^3}\int_0^l y^2 dy$$

$$= \frac{I_1\omega^2}{2l^3} \times \frac{l^3}{3}$$

$$= \frac{1}{3}\frac{1}{2} I_1\omega^2$$

$$= \frac{1}{3} \times \left[\frac{1}{2}(MOI \text{ of shaft}) \times (\text{angular velocity of free end})^2\right]$$

$$= \frac{1}{3} \times \text{KE of a disc of } MOI \text{ equal to that of the shaft}$$

attached to the free end of the shaft.

Thus to consider the inertia of the shaft, the moment of inertia of the disc is increased by an amount equal to one-third of that of the shaft.

Then

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I + \frac{I_1}{3}}} \tag{18.60}$$

18.22 MULTIFILAR SYSTEMS

Multifilar systems are used to determine the moment of inertia of irregular bodies such as unsymmetrical castings, spoked flywheels, connecting rods, etc., for which it is quite difficult to find their moment of inertia from their dimensions.

(i) Bifilar Suspension

Figure 18.34 represents a disc of mass m suspended from a rigid support with the help of two cords.

Let l = length of each cord

a and b = distance of centre of mass of the disc from the points of suspension of cords 1 and 2 respectively.

If the disc is now turned through a small angle θ about a vertical axis through the centre of mass, the cords will inclined to the vertical. On release, the disc will oscillate about the vertical axis and execute a torsional vibration.

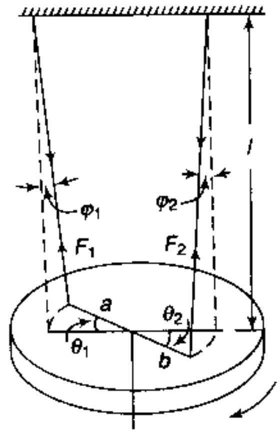


Fig. 18.34

Let θ = angular displacement of disc

φ_1, φ_2 = inclination of cords to the vertical

F_1, F_2 = tensions in cords 1 and 2 respectively

In the oscillating position, the effects of vertical accelerations can be neglected as the angles φ_1 and φ_2 are small and the torque produced can be considered to be only due to horizontal torques.

Restoring torque = (horizontal force on cord 1 × a) + (horizontal force on cord 2 × b)

$$= -[F_1 a \sin\varphi_1 + F_2 b \sin\varphi_2]$$

$$= -\{F_1 a\,\varphi_1 + F_2 b\,\varphi_2\} \qquad \text{(as } \varphi_1 \text{ and } \varphi_2 \text{ are small)}$$

$$= -\left[\frac{Wb}{a+b}a\varphi_1 + \frac{Wa}{a+b}b\varphi_2\right] \qquad (W = \text{Weight of disc} = mg)$$

$$= -\frac{Wab}{a+b}(\varphi_1 + \varphi_2)$$

$$= -\frac{Wab}{a+b}\left(\frac{a\theta}{l} + \frac{b\theta}{l}\right) \qquad (\varphi_1 l = a\theta \text{ and } \varphi_2 l = b\theta)$$

$$= -\frac{Wab}{(a+b)} \times \frac{\theta(a+b)}{l}$$

$$= -\frac{mgab}{l}\theta$$

Inertia torque $= -I\ddot{\theta} = -mk^2\ddot{\theta}$

Where k is the radius of gyration of the disc about the vertical axis through the centre of mass.

For equilibrium,

Inertia torque + Restoring torque = 0

$$mk^2\ddot{\theta} + \frac{mgab}{l}\theta = 0$$

or

$$\ddot{\theta} + \frac{gab}{lk^2}\theta = 0$$

∴
$$f_n = \frac{1}{2\pi}\sqrt{\frac{gab}{lk^2}}$$

or

$$k = \frac{1}{2\pi f_n}\sqrt{\frac{gab}{l}} \tag{18.61}$$

Thus radius of gyration can be found out by finding the natural frequency of vibration of the body.

(ii) Trifilar Suspension

Consider a disc of mass m (weight W), suspended by three vertical cords, each of length l, from a fixed support as shown in Fig. 18.35. Each cord is symmetrically attached to the disc at the same distance from the centre of mass of the disc.

If the disc is now turned through a small angle about its vertical axis, the cords become inclined. On being released, the disc will perform oscillations about the vertical axis. At any instant

let θ = angular displacement of the disc

φ = inclination of the cords to the vertical

F = tension in each cord = $W/3$

Inertia torque = $-I\ddot{\theta} = -mk^2\ddot{\theta}$

Restoring torque = $-3 \times$ (Horizontal component of force in each string $\times r$)

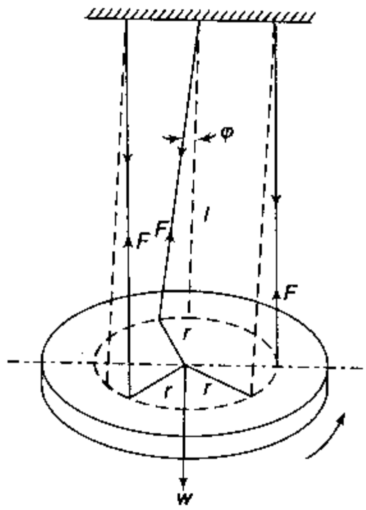


Fig. 18.35

$$= -3 \times Fr\sin\varphi$$

$$= -3Fr\varphi \quad \text{(as } \varphi \text{ is small)}$$

$$= -3Fr\frac{\theta r}{l} \quad (\because \varphi l = \theta r)$$

$$= -\frac{3W}{3} \times \frac{r^2}{l}\theta$$

$$= -\frac{mgr^2}{l}\theta$$

$$mk^2\ddot{\theta} + \frac{mgr^2}{l}\theta = 0$$

$$\ddot{\theta} + \frac{gr^2}{lk^2}\theta = 0$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{gr^2}{lk^2}}$$

$$k = \frac{r}{2\pi f_n}\sqrt{\frac{gr^2}{l}} \tag{18.62}$$

Example 18.21 *Determine the frequency of torsional vibrations of the disc shown in Fig. 18.36 if both the ends of the shaft are fixed and the diameter of the shaft is 40 mm. The disc has a mass of 96 kg and a radius of gyration of 0.4 m. Take modulus of rigidity for the shaft material as 85 GN/m². l_1 = 1m and l_2 = 0.8 m.*

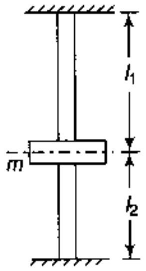


Fig. 18.36

Solution

$m = 96$ kg $G = 85 \times 10^9$ N.m²

$k = 0.4$ m $d = 0.04$ m

$$I = mk^2 = 96 \times (0.4)^2$$
$$= 15.36 \text{ kg.m}^2$$
$$J = \frac{\pi}{32}d^4 = \frac{\pi}{32} \times (0.04)^4$$
$$= 0.251 \times 10^{-6} \text{ m}^4$$

Total torsional stiffness of shaft, $q = q_1 + q_2$

$$= \frac{GJ}{l_1} + \frac{GJ}{l_2}$$
$$= 85 \times 10^9 \times 0.251 \times 10^{-6}\left(\frac{1}{1} + \frac{1}{0.8}\right)$$
$$= 48\,004 \text{ N.m}$$
$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{48\,004}{15.36}}$$
$$= \underline{8.9 \text{ Hz}}$$

Example 18.22 *Determine the natural frequency of a simple pendulum (Fig. 18.37), taking the mass of the rod into consideration.*

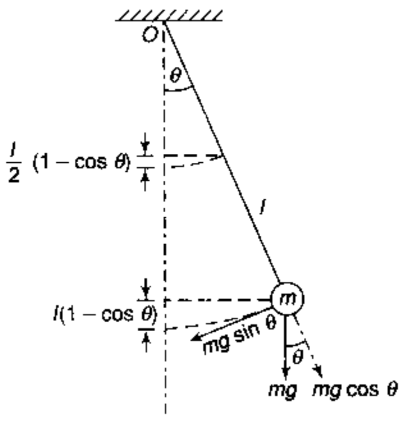


Fig. 18.37

Solution

Equilibrium Method

Taking moments about θ,

$$I_0\ddot{\theta} + mg(l\sin\theta) + m_r g\left(\frac{l}{2}\sin\theta\right) = 0$$
$$\left(ml^2 + \frac{m_r}{3}l^2\right)\ddot{\theta} + mgl\theta + m_r gl\frac{\theta}{2} = 0 \quad (\theta \text{ is small})$$
$$\left(m + \frac{m_r}{3}\right)l^2\ddot{\theta} + gl\left(m + \frac{m_r}{2}\right)\theta = 0$$
$$\ddot{\theta} + \frac{g}{l}\left(\frac{m + \frac{m_r}{2}}{m + \frac{m_r}{3}}\right)\theta = 0$$
$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}\frac{m + (m_r/2)}{m + (m_r/3)}}$$

Energy Method At any instant,

$$\frac{d}{dt}(KE + PE) = 0$$

or

$$\frac{d}{dt}\left[\frac{1}{2}I_0\dot{\theta}^2 + \left\{\begin{matrix} mgl(1-\cos\theta) \\ +m_r g\frac{1}{2}(1-\cos\theta)\end{matrix}\right\}\right] = 0$$

or

$$\frac{d}{dt}\left[\begin{matrix}\frac{1}{2}\left(m + \frac{m_r}{3}\right)l^2\dot{\theta}^2 + gl \\ \left(m + \frac{m_r}{3}\right)(1-\cos\theta)\end{matrix}\right] = 0$$

or

$$\frac{1}{2}\left(m+\frac{m_r}{3}\right)l^2(2\dot{\theta}\ddot{\theta})+gl$$

$$\left(m+\frac{m_r}{3}\right)\sin\theta\dot{\theta}=0$$

or

$$\left(m+\frac{m_r}{3}\right)l\ddot{\theta}+g\left(m+\frac{m_r}{3}\right)\theta=0\ldots$$

(θ is small)

or

$$\ddot{\theta}+\frac{g}{l}\left(\frac{m+\frac{m_r}{2}}{m+\frac{m_r}{3}}\right)\theta=0$$

$$f_n=\frac{1}{2\pi}\sqrt{\frac{g}{l}\frac{m+\frac{m_r}{2}}{m+\frac{m_r}{3}}}$$

If the mass of the rod is neglected,

$$f_n=\frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

Example 18.23 ***Find the natural frequency of the oscillation in the cases shown in Fig. 18.38(a) and (b). The roller rolls on the surface without slipping.***

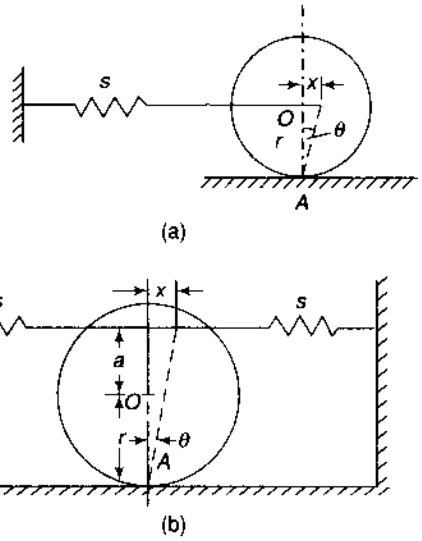


Fig. 18.38

Solution

(a) *Equilibrium (Newton's Method)*

Taking moments about the instantaneous centre A, considering small oscillations of the disc,

$$I_a\ddot{\theta}+(sx)r=0$$

or $(I_0+mr^2)\ddot{\theta}+(s\,\theta\,r)\,r=0$

or $\left(\frac{1}{2}mr^2+mr^2\right)\ddot{\theta}+sr^2\theta=0$

or $\ddot{\theta}+\frac{sr^2}{\frac{3}{2}mr^2}\theta=0$

or $\ddot{\theta}+\frac{2s}{3m}\theta=0$

$$f_n=\frac{1}{2\pi}\sqrt{\frac{2s}{3m}}\text{ Hz}$$

Energy Method

$$\frac{d}{dt}(KE+PE)=0$$

$$\frac{d}{dt}\left[\frac{1}{2}I_a\dot{\theta}^2+\frac{1}{2}sx^2\right]=0$$

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot{\theta}^2+\frac{1}{2}s(\theta r)^2\right]=0$$

$$\frac{d}{dt}\left[\frac{3}{4}mr^2\dot{\theta}^2+\frac{1}{2}sr^2\theta^2\right]=0$$

$$\frac{3}{4}mr^2\times 2\dot{\theta}\ddot{\theta}+\frac{1}{2}sr^2\times 2\theta\dot{\theta}=0$$

$$\ddot{\theta}+\frac{2s}{3m}\theta=0$$

i.e., the same equation as before.

(b) *Newton's Method*

Taking moments about A.

$I_a\ddot{\theta}+2(sx)(r+a)=0$ (there are two springs)

$(I_0+mr^2)\ddot{\theta}+2s\,[(r+a)\theta](r+a)=0$

or $\left(\frac{1}{2}mr^2+mr^2\right)\ddot{\theta}+2s(r+a)^2\theta=0$